MAT LANGFORD

FRATERNAL TWINS

THE UNREASONABLE RESEMBLANCE OF THE RICCI AND MEAN CURVATURE FLOWS

Cover by Mat Langford. Photograph of Gerhard Huisken by Gerd Fischer. Photograph of Richard Hamilton by George M. Bergman. Photographs courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach. Illustrations by Mat Langford.

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Dedicated to Richard S. Hamilton
1943–2024

"Like leaves on trees the race of man is found,
Now green in youth, now withering on the ground;
Another race the following spring supplies;
They fall successive, and successive rise:
So generations in their course decay;
So flourish these, when those are pass'd away."

– *Homer,* The Iliad (6:171), *tr. Alexander Pope.*

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Preface

On June 4, 2019, at a conference hosted by the ETH Zürich on geometric analysis and general relativity in honour of Gerhard Huisken's 60th birthday, Richard Hamilton presented a lecture enigmatically entitled "Fraternal Twins". In this lecture, he presented an overview of the key historical and mathematical developments in the study of the mean curvature and Ricci flows, emphasizing the striking similarities which consistently occur at a superficial level, but also pointing out the imperfection of these similarities, and some of the analytical differences which lie behind them—much like fraternal twins, the two flows appear very alike at first sight, even though they are by no means identical.

Recognition of the likeness of the two flows goes back much further, of course. Indeed, the drawing of parallels between the two flows is now customary amongst experts; it is often exclaimed, for instance, that "Ricci flow is the extrinsic analogue of mean curvature flow", or that "mean curvature flow in n-dimensions behaves like Ricci flow in $2n$-dimensions", or "since P holds for mean curvature flow/Ricci flow, $\tilde{P}$ *must* be true for Ricci flow/mean curvature flow".[1] And the comparison is more than superficial: despite the fact that the two flows continue to be treated independently, often with quite different tools, Hamilton's analogy continues to be vindicated.

The aim of this book is to provide an introduction to geometric evolution equations through a study of these twin flows. It contains two parts: the first is dedicated to the mean curvature flow and the second to the Ricci flow, though the order does not matter much: each part may be treated entirely independently of the other. On the other hand, once the reader has gained some familiarity with one twin, they will feel at once an uncanny familiarity with the other.

We do not attempt to provide a comprehensive treatment[2] of our twin subjects but rather offer the reader an enticing *aperitif*, which we hope may whet their appetite for the subject.[3] Each part begins with *The fundamentals*, introducing the reader to each twin, followed by a technical chapter which lays *The groundwork* for further analysis. This second chapter could be skipped on first reading, and referred

[1] Some brave souls even speculate that there is a hidden canonical correspondence between the two; but no such correspondence is yet to be observed.

[2] This would take up many volumes, and has already been achieved, to a large degree, by others.

[3] Incidentally, we heartily recommend a glass of *Glenlivet (Founder's Reserve)* to accompany this text, not least because "Glenlivet" may be translated as "Valley of the smooth flow".

back to as needed in the later chapters; on the other hand, the patient reader will certainly benefit in the long run from any effort put into the groundwork. The third chapter of each part is concerned with curvature *Pinching and its consequences*, with a focus on the first major milestone in each of our twin subjects—Huisken's theorem on the contraction of convex hypersurfaces to round points under mean curvature flow and Hamilton's theorem on the contraction of three-manifolds of positive Ricci curvature to round points under Ricci flow, respectively. We then study each flow in its smallest nontrivial dimension, where the behaviour is particularly nice. The fifth chapter introduces the reader to a selection of tools and results pertaining to *Singularities and their analysis* for the respective flow (in higher dimensions). We conclude by surveying some of the recent progress *Towards a classification of ancient solutions* to each flow.

Each chapter ends with a selection of exercises, and the book would be well-suited to a one or two semester graduate course in geometry, or even an undergraduate "special topics" course. For a one semester course, one could plausibly cover, e.g., Chapters 1-5, or Chapters 7-11, or selected parts of Chapters 1-4 and 7-10[4].

[4] A great deal of material can be covered by adopting an alternating structure—1,7,2,8,3,9,...—due to much constructive *approximate* redundancy arising from the fraternal resemblance of the two subjects.

The project grew out of notes for a minicourse on the Ricci flow which I presented in a series of lectures at the summer school "Geometric Flows and Relativity" hosted by the Centro de Matemática of the Universidad de la República in Montevideo, Uruguay, in March 2024, which were subsequently used in a special topics course on both the mean curvature and Ricci flows aimed at advanced undergraduate and beginning graduate students at The Australian National University. I am grateful to Theodora Bourni and Martín Reiris for the invitation to speak at the CMAT summer school, and to the outstanding cohort of students who attended my lectures, keeping me on my toes each morning; I am equally grateful to my wife, Kirsty, who—heavily pregnant with our second child—encouraged me to go!

Many individuals have contributed to this book through useful discussions, particularly Ben Andrews, Theodora Bourni, Tim Buttsworth, Bennett Chow, Apostolos Damialis, Ramiro Lafuente, Stephen Lynch, Martín Reiris and Jonathan Zhu.

I do not claim priority for any of the mathematical results presented herein, and have endeavoured to provide appropriate bibliographic information throughout. The manuscript was compiled on Overleaf in Tufte-LaTeX and the cover was designed using Adobe Illustrator and Adobe Express. Illustrations were created using GeoGebra and Mathematica. No AI tools were used in any stage of the preparation.

Mat Langford
Canberra, March 28, 2025

11 Conformal flow of surfaces by curvature

A key step in the proof of Hamilton's theorem on the convergence of three-manifolds of positive Ricci curvature (and its higher dimensional analogues) was the improvement of pinching of the eigenvalues of the Ricci curvature (or curvature operator). No such estimate is possible in the two-dimensional setting as, in that case, the curvature operator has only one component! Fortunately, in two-dimensions, the Ricci flow enjoys some additional structure, which actually allows us to prove something far stronger.

11.1 Special properties of the Ricci flow in two space dimensions

Since in two dimensions the Ricci tensor is in proportion to the metric[1], the Ricci flow takes the form

$$\partial_t g = -2\mathrm{K} g, \tag{11.1}$$

where the scalar of proportion, K, is called the GAUSS CURVATURE. This equation is also the two-dimensional special case of a number of other higher dimensional flows (e.g. the Kähler Ricci flow, the Yamabe flow, and conformal flows by functions of the Schouten tensor). With this in mind, it is perhaps not surprising that (11.1) displays properties of these higher dimensional flows that are not necessarily shared by the Ricci flow in higher dimensions.

[1] This is a straightforward consequence of the algebraic symmetry properties of the Riemann curvature tensor.

11.1.1 The logarithmic fast diffusion equation and conformal invariance

Two dimensional Ricci flow $(M^2 \times I, g)$ of a compact manifold M^2 is actually a CONFORMAL FLOW; that is, we can find a function $u \in C^\infty(M^2 \times I)$ such that

$$g_{(x,t)} = \mathrm{e}^{-2u(x,t)} g_{(x,0)}\,. \tag{11.2}$$

To prove this, observe that a time-dependent metric of the form (11.2) satisfies Ricci flow if and only if

$$\partial_t u g = -\frac{1}{2}\mathcal{L}_{\partial_t} g = \mathrm{Rc} = \mathrm{K} g.$$

That is,

$$\partial_t u = \mathrm{K}.$$

By Exercise 11.1,

$$\mathrm{K}(x,t) = \mathrm{e}^{2u(x,t)}(\Delta_0 u(x,t) + \mathrm{K}_0(x)),$$

where Δ_0 and K_0 are the Laplace–Beltrami operator and sectional curvature of g_0, so we conclude that $\mathrm{e}^{-2u} g_0$ satisfies Ricci flow if and only if

$$\partial_t u = \mathrm{e}^{2u}(\Delta_0 u + \mathrm{K}_0). \tag{11.3}$$

But this is a parabolic equation, and hence admits a (unique) solution u for a short-time, given the initial condition $u_0 = 0$. By uniqueness of solutions to Ricci flow on compact manifolds, $g = \mathrm{e}^{-2u} g_0$ must be the unique Ricci flow starting from g_0.

We note that (11.3) is equivalent to the LOGARITHMIC FAST DIFFUSION EQUATION

$$\partial_t v = \Delta_0 \log v - 2\mathrm{K}_0 \tag{11.4}$$

on (M^2, g_0) for the conformal factor $v = \mathrm{e}^{-2u}$.

11.1.2 *Preservation of negative curvature*

Since $\mathrm{Rc} = \mathrm{K} g$, the Gauss curvature (which is half the scalar curvature) evolves according to

$$(\partial_t - \Delta)\mathrm{K} = 2\mathrm{K}^2. \tag{11.5}$$

This means that negativity of curvature is preserved in two dimensions (recall that positivity of the scalar curvature is preserved in all dimensions). We also obtain an analogue of Proposition 9.11:

Proposition 11.1. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a compact two-manifold M^2.*

1. *If* $\max_{M^2 \times \{\alpha\}} \mathrm{K} = 0$ *then either* $\mathrm{K} \equiv 0$ *or* $\mathrm{K} < 0$ *for* $t \in (\alpha, \omega)$.
2. *If* $\max_{M^2 \times \{\alpha\}} \mathrm{K} = -r^{-2} < 0$, *then*

$$\max_{M^2 \times \{t\}} \mathrm{K} \leq -\frac{1}{r^2 + 2(t-\alpha)}$$

for $t \in (\alpha, \omega)$.

3. *If* $\max_{M^2\times\{\alpha\}} \mathrm{K} = r^{-2} > 0$*, then*

$$\max_{M^2\times\{t\}} \mathrm{K} \le \frac{1}{r^2 - 2(t-\alpha)}$$

for $t \in (\alpha, \omega)$*.*

In fact, we can do better by making use of the Gauss–Bonnet theorem.

11.1.3 *A simple formula for the area*

By the Gauss–Bonnet theorem and the first variation of area, the area of a two-dimensional Ricci flow changes at a precise rate:

$$\frac{d}{dt}\,\mathrm{area}(t) = -2\int_{M^2} \mathrm{K}\, d\mu = -4\pi\chi(M^2), \tag{11.6}$$

where $\chi(M^2)$ is the Euler characteristic of M^2. Integrating yields

$$\mathrm{area}(M^2, t) = \mathrm{area}(M^2, 0) - 4\pi\chi(M^2)t, \tag{11.7}$$

a remarkably simple (and useful) formula. Indeed, consider the average Gauss curvature

$$\kappa(t) \doteqdot \frac{\int_{M^2} \mathrm{K} d\mu}{\int_{M^2} d\mu} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2, t)} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2, 0) - 4\pi\chi(M^2)t}.$$

By (11.6) (or (11.7)),

$$\frac{d}{dt}\kappa = -\frac{2\pi\chi(M^2)}{\mathrm{area}^2(M^2, t)}\frac{d}{dt}\,\mathrm{area}(M^2, t) = 2\kappa^2.$$

Recalling (11.5), we thus find that

$$(\partial_t - \Delta)(\mathrm{K} - \kappa) = 2(\mathrm{K} - \kappa)\left(\mathrm{K} - \kappa + \frac{4\pi\chi(M^2)}{\mathrm{area}(M^2, 0) - 4\pi\chi(M^2)t}\right)$$

and hence, if we normalize so that $\mathrm{area}(M^2, 0) = 4\pi$,

$$\min_{M^2\times\{t\}} \mathrm{K} \ge \kappa + \phi \tag{11.8}$$

for $t \in [0, T)$, where ϕ is the solution to the problem

$$\begin{cases} \dfrac{d\phi}{dt} = 2\phi\left(\phi + \dfrac{\chi(M^2)}{1 - \chi(M^2)t}\right) \\ \phi(0) = \phi_0 \doteqdot \displaystyle\min_{M^2\times\{0\}} (\mathrm{K} - \kappa)\,; \end{cases}$$

that is (note that $\phi_0 \le 0$),

$$\phi(t) = \frac{\phi_0}{(1-\chi(M^2)t)(1-\chi(M^2)t-2\phi_0 t)}$$
$$\sim \begin{cases} -\frac{1}{t^2} \text{ as } t\to\infty \text{ if } \chi(M^2)<0 \\ -\frac{1}{t} \text{ as } t\to\infty \text{ if } \chi(M^2)=0 \\ \frac{-1}{1-\chi(M^2)t} \text{ as } t\to\frac{1}{\chi(M^2)} \text{ if } \chi(M^2)>0. \end{cases}$$

In particular, $T \le \frac{1}{\chi(M^2)}$ if $\chi(M^2) > 0$.

An analogous argument may be carried out to establish an upper bound for $\max_{M^2\times\{t\}} \mathrm{K}$, but that estimate will prove of little utility. We will obtain a congruous estimate from above by a different argument, which is strongly informed by the behaviour of solitons.

11.1.4 The Chow–Hamilton entropy

The Chow–Hamilton entropy[2] of a Riemannian surface $(M^2, g)$of positive curvature is defined to be

$$\mathscr{E}(M^2,g) \doteqdot \frac{\operatorname{area}(M^2,g)}{\chi(M^2)}\exp\left(\frac{1}{\chi(M^2)}\int_{M^2}\mathrm{K}\log\mathrm{K}\,d\mu\right). \qquad (11.9)$$

[2] Compare this to the Nash entropy, $-\int u\log u$, of a positive function u, introduced by Nash, "Continuity of solutions of parabolic and elliptic equations".

Proposition 11.2 (Monotonicity of the Chow–Hamilton entropy[3]). *Along any Ricci flow with positive curvature $(M^2\times I, g)$ on a compact surface M^2,*

$$\frac{d}{dt}\mathscr{E}(M^2,g_t) \le 0$$

at all times, with strict inequality unless $\partial_t \log\mathrm{K} - |\nabla\log\mathrm{K}|^2$ is constant in space.

[3] The stated result was established by Richard S. Hamilton, "The Ricci flow on surfaces". A modified version which allows the curvature to change sign was established by Chow, "The Ricci flow on the 2-sphere".

Proof. To make the calculations slightly simpler, we assume that $M^2 \cong S^2$ but the general case is the same. Using the evolution equations for curvature (11.5) and area (9.8), we find that

$$\frac{d}{dt}\int_{M^2}\mathrm{K}\log\mathrm{K}\,d\mu = \int_{M^2}\left(\frac{\partial_t\mathrm{K}}{\mathrm{K}} - \frac{|\nabla\mathrm{K}|^2}{\mathrm{K}^2}\right)\mathrm{K}\,d\mu.$$

Set

$$Q \doteqdot \partial_t\log\mathrm{K} - |\nabla\log\mathrm{K}|^2.$$

Using the formulae

$$[\nabla,\Delta]f = -\mathrm{K}\nabla f,\ \ \nabla_t\nabla f = \nabla\partial_t f + \mathrm{K}\nabla f \ \text{ and } \ [\partial_t,\Delta]f = 2\mathrm{K}\Delta$$

(and a little elbow grease) we find that

$$(\partial_t - \Delta)Q = 2g(\nabla\log\mathrm{K},\nabla Q) + 2\left|\nabla^2\log\mathrm{K} + \mathrm{K}g\right|^2. \qquad (11.10)$$

We thus find that

$$\begin{aligned}\frac{d^2}{dt^2}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu &= \frac{d}{dt}\int_{M^2} Q\mathrm{K}\, d\mu \\ &= \int_{M^2} \left(\mathrm{K}\,\partial_t Q + Q\Delta\,\mathrm{K}\right)\, d\mu \\ &= \int_{M^2} \left(\Delta(\mathrm{K}Q) + 2\,\mathrm{K}\left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2\right)\, d\mu \\ &= 2\int_{M^2} \left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2 \mathrm{K}\, d\mu .\end{aligned}$$

Estimating

$$\begin{aligned}\left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2 &= \left|\nabla^2\log\mathrm{K} - \tfrac{1}{2}\Delta\log\mathrm{K}\,g + \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)g\right|^2 \\ &= \left|\nabla^2\log\mathrm{K} - \tfrac{1}{2}\Delta\log\mathrm{K}\,g\right|^2 + \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)^2 \\ &\geq \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)^2 \\ &= \tfrac{1}{2}Q^2, \end{aligned} \tag{11.11}$$

this becomes

$$\frac{d^2}{dt^2}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu \geq \int_{M^2} Q^2\,\mathrm{K}\, d\mu .$$

Applying Hölder's inequality and the Gauss–Bonnet theorem, we arrive at

$$\frac{d^2}{dt^2}\int_{\Gamma_t} \mathrm{K}\log\mathrm{K}\, ds \geq \frac{1}{4\pi}\left(\frac{d}{dt}\int_{\Gamma_t} \mathrm{K}\log\mathrm{K}\, ds\right)^2 . \tag{11.12}$$

On the other hand, recalling (4.2), we see that the function

$$\phi(t) \doteqdot \frac{32\pi^2}{\mathrm{area}(M^2, g_t)} = \frac{4\pi}{\frac{\mathrm{area}(M^2, g_0)}{8\pi} - t}$$

satisfies the corresponding ODE

$$\frac{d\phi}{dt} = \tfrac{1}{4\pi}\phi^2 .$$

Moreover, by Perelman's curvature estimate (Theorem 9.21), the flow may be continued until the area tends to zero[4]; i.e. (by (9.8)) until time $T \doteqdot \frac{\mathrm{area}(M^2, g)}{8\pi}$. This means that

[4] We shall present an alternative argument for this below.

$$\phi(t) \to \infty \ \text{ as } \ t \to T ,$$

and we may thereby deduce, by ODE comparison, that

$$\begin{aligned}\frac{d}{dt}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu &\leq \frac{32\pi^2}{\mathrm{area}(M^2, g_t)} \\ &= -4\pi\frac{d}{dt}\log\mathrm{area}(M^2, g_t) .\end{aligned}$$

Rearranging, we conclude that

$$\frac{d}{dt}\log \mathscr{E}(\Gamma_t) \le 0.$$

Now, if the inequality is saturated at some time t_0, then we may deduce from (11.12) that is saturated for all $t \le t_0$. But this guarantees that the Hölder inequality is saturated, which ensures that Q is constant in space for $t \le t_0$. □

11.2 Self-similar solutions

Recall that a metric g on a two-manifold M^2 generates a self-similarly expanding, steady or shrinking Ricci flow if there are a constant $\lambda \in \mathbb{R}$ and a vector field V such that

$$\mathrm{Rc} = \lambda g - \tfrac{1}{2}\mathcal{L}_V g. \tag{11.13a}$$

An important special class of solutions are those with $V = \operatorname{grad} f$ for some POTENTIAL FUNCTION f, in which case,

$$\mathrm{Rc} = \lambda g - \nabla^2 f. \tag{11.13b}$$

Theorem 11.3. *Every compact, two-dimensional gradient Ricci soliton has constant curvature.*

Proof. Let (M^2, g, f) be a gradient Ricci soliton on a compact two-manifold. By Exercise 11.2, the vector field $K \doteqdot \mathrm{J}(\nabla f)$ is Killing. Since M^2 is compact, there must be some $o \in M^2$ such that $\nabla f(o) = 0$ and hence $K(o) = 0$. It follows that K generates rotations, and hence we can find coordinates $(r, \theta) \in (0, R) \times \mathbb{R}/2\pi\mathbb{Z}$ such that $g = dr^2 + \psi^2(r)d\theta^2$. The claim now follows from the result of Exercise 8.1. □

Essentially the same argument yields the following.

Theorem 11.4. *The cigar is the only steady two-dimensional gradient Ricci soliton with positive curvature.*

Sketch of the proof. By Theorem 11.3, M^2 cannot be compact. It follows from Theorem 9.21 (though indirectly; see Theorem 13.2 below) that $\mathrm{K} \to 0$ as the distance to any fixed point x of M^2 goes to infinity. But then K attains a (positive) maximum at some point, at which $\nabla f = \nabla \mathrm{K}/\mathrm{K} = 0$. The claim now follows as in the previous theorem and Example 17. □

By manipulating the (gradient) soliton equation, we shall establish a suite of identities for two-dimensional (gradient) Ricci solitons.

First observe that taking the trace of (11.13a) yields (note that, for any vector field V, $\frac{1}{2}\mathcal{L}_V g$ is equal to the symmetric part of ∇V)

$$\mathrm{K} = \lambda - \tfrac{1}{2}\operatorname{div} V, \tag{11.14a}$$

or, in the gradient case,

$$-\Delta f = 2(\mathrm{K} - \kappa). \tag{11.14b}$$

From this, we see that (11.13a) is equivalent to

$$\mathcal{L}_V g - \operatorname{div} V g = 0. \tag{11.15a}$$

or, in the gradient case,

$$\nabla^2 f - \tfrac{1}{2}\Delta f g = 0. \tag{11.15b}$$

Moreover, in case M^2 is compact,

$$0 = -\int_{M^2} \operatorname{div} V \, d\mu = 2\int_{M^2} (\mathrm{K} - \lambda)\, d\mu$$

and hence

$$\lambda = \kappa \doteqdot \frac{\int_{M^2} \mathrm{K}\, d\mu}{\int_{M^2} d\mu}.$$

Taking the divergence of (11.15a), we find that

$$\Delta V + \mathrm{Rc}(V) = 0 \tag{11.16a}$$

which, on a gradient Ricci soliton becomes

$$\nabla \mathrm{K} - \mathrm{K}\nabla f = 0. \tag{11.16b}$$

Next observe that taking the divergence of (11.16b) yields, in the gradient case,

$$\Delta \mathrm{K} - \nabla_{\nabla f}\mathrm{K} + 2\mathrm{K}(\mathrm{K} - \lambda) = 0. \tag{11.17}$$

We may also rewrite (11.16b), using (11.15b), as

$$\begin{aligned} 0 &= \nabla \mathrm{K} - (\mathrm{K} - \lambda)\nabla f - \lambda \nabla f \\ &= \nabla \mathrm{K} + \tfrac{1}{2}\Delta f \nabla f - \lambda \nabla f \\ &= \nabla \mathrm{K} + \nabla_{\nabla f}\nabla f - \lambda \nabla f \\ &= \nabla\left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \lambda f\right). \end{aligned} \tag{11.18}$$

Thus, in the gradient case,

$$\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \lambda f = C$$

for some constant $C \in \mathbb{R}$. Equivalently (by (11.14b)),

$$-\Delta f + \tfrac{1}{2}|\nabla f|^2 - \mathrm{K} - \lambda f = C - 2\lambda.$$

Remarkably, this is the Euler–Lagrange equation for a certain constrained energy functional.

Proposition 11.5. *Given any compact Riemannian surface (M^2, g) and any $\lambda \in \mathbb{R}$, define, for any smooth function f,*

$$F(f) \doteqdot \int_{M^2} \left(\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) \mathrm{e}^{-f}\, d\mu. \tag{11.19}$$

If $\{f_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a smooth variation of $f = f_0$ which satisfies the weighted volume constraint

$$\frac{d}{d\varepsilon}\int_{M^2} \mathrm{e}^{-f_\varepsilon}\, d\mu \equiv 0,$$

then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(f_\varepsilon) = -\int_{M^2} \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) h\, \mathrm{e}^{-f}\, d\mu,$$

where $h \doteqdot \frac{d}{d\varepsilon}|_{\varepsilon=0} f_\varepsilon$. In particular, if f is a stationary point of the action with respect to constrained variations, then $-\Delta f + \frac{1}{2}|\nabla f|^2 - \mathrm{K} - \lambda f$ is constant.

Proof. Since the weighted volume constraint guarantees that

$$0 = -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{M^2} \mathrm{e}^{-f_\varepsilon}\, d\mu = \int_{M^2} h\mathrm{e}^{-f}\, d\mu,$$

we find that

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(f_\varepsilon) &= \int_{M^2} \left(g(\nabla f, \nabla h) + \lambda h - h\left[\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right]\right) \mathrm{e}^{-f}\, d\mu \\ &= -\int_{M^2} \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) h\mathrm{e}^{-f}\, d\mu.\end{aligned}$$

This is the first claim. The second claim follows since any function h which is $L^2(\mathrm{e}^{-f}\, d\mu)$-orthogonal to the constant functions gives rise to an admissible variation. □

Theorem 11.6. *All compact, two-dimensional shrinking Ricci solitons are gradient.*

Sketch of the proof. We will prove this statement in all dimensions in §12.2. The idea is to find a minimizer f for the functional F (using classical methods from the calculus of variations). If $\lambda g - \mathrm{Rc} = \frac{1}{2}\mathcal{L}_V g$ for some vectorfield V (the Ricci soliton equation), then this minimizer will satisfy $\lambda g - \mathrm{Rc} = \nabla^2 f$ (the *gradient* Ricci soliton equation). (Note that this does not necessarily mean that the original soliton vector field V is given by $V = \nabla f$—the two could differ by a Killing vector field). □

Combined with Theorem 11.3, we find that

Corollary 11.7. *every compact, two-dimensional shrinking Ricci soliton has constant curvature.*

Consider now, for some gradient Ricci soliton (M^2, g, f), the corresponding self-similar Ricci flow $\phi^* g$, ϕ being the flow of ∇f. This Ricci flow will satisfy the soliton equation with

$$\lambda(t) = \begin{cases} \dfrac{1}{-2t} \text{ for } t \in (-\infty, 0) \text{ (shrinking case)} \\ 0 \text{ for } t \in (-\infty, \infty) \text{ (steady case)} \\ \dfrac{1}{2t} \text{ for } t \in (0, \infty) \text{ (expanding case).} \end{cases}$$

Thus, by (11.18),

$$\begin{aligned} \partial_t f &= \nabla_{\nabla f} f \\ &= |\nabla f|^2 \\ &= -2\,\mathrm{K} + 2\lambda f + C \\ &= -2(\mathrm{K} - \lambda) - 2\lambda + 2\lambda f + C \\ &= \Delta f + 2\lambda f + C - 2\lambda. \end{aligned}$$

Since we are free to modify the potential function, at each time, by addition of a constant, some choice of potential function will satisfy the heat equation

$$(\partial_t - \Delta) f = 2\lambda f. \tag{11.20}$$

Alternatively, since $-\Delta f = 2(\mathrm{K} - \lambda)$, we may exhibit f as a solution to the *backwards* heat equation

$$\begin{aligned} (\partial_t + \Delta) f + 2\,\mathrm{K} &= 2\Delta f + 2\lambda f + C + 2(\mathrm{K} - \lambda) \\ &= -2(\mathrm{K} - \lambda) + 2\lambda f + C \\ &= |\nabla f|^2 + 2\lambda. \end{aligned} \tag{11.21}$$

Remarkably, this means that the function $h \doteqdot \lambda e^{-f}$ satisfies the CONJUGATE HEAT EQUATION:

$$-(\partial_t + \Delta - 2\mathrm{K}) h = 0.$$

The name comes from the fact that, along any two-dimensional Ricci flow $(M^2 \times I, g)$,

$$\begin{aligned} \frac{d}{dt} \int_{M^2} u\varphi \, d\mu &= \int_{M^2} \left(\partial_t u \varphi + u \partial_t \varphi - 2\mathrm{K} u \varphi \right) d\mu \\ &= \int_{M^2} \left((\partial_t - \Delta) u \varphi + u (\partial_t + \Delta - 2\mathrm{K}) \varphi \right) d\mu. \end{aligned}$$

so long as $\varphi(\cdot, t)$ is compactly supported. In particular, a smooth function $u : M^2 \times (a, b) \to \mathbb{R}$ satisfies the heat equation if and only if every smooth function $\varphi : M^2 \times (a, b) \to \mathbb{R}$ which is compactly supported in $M^2 \times (a, b)$ satisfies

$$\int_a^b \int_{M^2} u (\partial_t - \Delta)^* \varphi \, d\mu \, dt = 0.$$

where

$$(\partial_t - \Delta)^* = -(\partial_t + \Delta - 2\mathrm{K})$$

is the CONJUGATE HEAT OPERATOR.

11.3 The differential Harnack inequality

The classical heat equation exhibits a remarkable property, known as the (matrix) DIFFERENTIAL HARNACK INEQUALITY, which states that any positive solution $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ must satisfy

$$\nabla^2 \log u + \frac{\mathrm{I}}{2t} \geq 0. \tag{11.22}$$

In fact, the inequality must be strict, unless u is a constant multiple of the (self-similar) fundamental solution, $\rho(x,t) \doteqdot (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-x_0|^2}{4t}}$ for some x_0. Integrating the trace of (11.22) along spacetime curves of the form $t \mapsto (\gamma(t), t)$, with γ a geodesic joining points x_1 and x_2, yields the classical HARNACK INEQUALITY:

$$(4\pi t_2)^{\frac{n}{2}} u(x_2,t_2) \geq (4\pi t_1)^{\frac{n}{2}} u(x_1,t_1) \exp\left(-\frac{|x_2-x_1|^2}{4(t_2-t_1)}\right), \tag{11.23}$$

for any x_2, x_1 and any $t_2 > t_1$.

For an ANCIENT[5] solution $u : \mathbb{R}^n \times (-\infty,\infty) \to \mathbb{R}$, performing a series of time-translations yields the stronger inequality

[5] I.e. having an infinite past.

$$\nabla^2 \log u \geq 0.$$

Again, we have strict inequality, except in the exceptional circumstance that $\nabla^2 \log u = 0$; that is, u is a constant multiple of the travelling wave solution, $u(x,t) = \mathrm{e}^{(x+tv)\cdot v}$ for some $v \in \mathbb{R}^n$.

Observe that, by (11.16b) and (11.17), a two-dimensional expanding gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}} - \frac{\mathrm{K}}{t},$$

while a two-dimensional steady gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}}.$$

Theorem 11.8 (Differential Harnack inequality[6]). *Along any Ricci flow $(M^2 \times [0,T), g)$ with positive curvature on a compact two-manifold,*

[6] Richard S. Hamilton, "The Ricci flow on surfaces". Cf. Chow, "The Ricci flow on the 2-sphere"

$$\frac{\partial_t \mathrm{K}}{\mathrm{K}} - \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}^2} + \frac{1}{t} \geq 0 \;\; \textit{for} \;\; t \in (0,T),. \tag{11.24}$$

Moreover, if (11.24) *holds along a Ricci flow $(M^2 \times (0,T), g)$ on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless $(M^2 \times (0,T), g)$ is an expanding self-similar solution.*

On any non-flat ancient two-dimensional Ricci flow $(M^2 \times (-\infty, T), g)$*,*

$$\frac{\partial_t K}{K} - \frac{|\nabla K|^2}{K^2} \geq 0. \tag{11.25}$$

Moreover, if (11.25) *holds along Ricci flow* $(M^2 \times (-\infty, T), g)$ *on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless* $(M^2 \times (-\infty, T), g)$ *is a steady self-similar solution.*

Proof. Consider the functions

$$Q \doteqdot \partial_t \log K - |\nabla \log K|^2$$

and

$$P \doteqdot t(\partial_t \log K - |\nabla \log K|^2) + 1.$$

Note that $P \equiv 0$ if and only if $(M^n \times I, g)$ is an expanding self-similar solution and $Q \equiv 0$ if and only if $(M^n \times I, g)$ is a steady self-similar solution.

Recalling (11.10), we have

$$(\partial_t - \Delta)Q = 2g(\nabla \log K, \nabla Q) + 2\left|\nabla^2 \log K + K\, g\right|^2.$$

Applying (11.11), we thus find that

$$(\partial_t - \Delta)P \geq 2g(\nabla \log K, \nabla P) + QP.$$

Since $P|_{t=0} = 1 > 0$, the maximum principle implies that $P \geq 0$ for positive times, and either $P > 0$ or $P \equiv 0$. The claims follow. □

Note that, by continuity, smooth limits of Ricci flows on compact surfaces satisfy the differential Harnack inequality (and hence also the rigidity case by the strong maximum principle).

Corollary 11.9 ((Integral) Harnack inequality)**.** *Along any Ricci flow* $(M^2 \times [0, T), g)$ *with positive curvature on a compact two-manifold,*

$$\frac{K(x_2, t_2)}{K(x_1, t_1)} \geq \left[\frac{t_2}{t_1} \exp\left(\frac{d^2(x_1, x_2, t_1)}{4(t_2 - t_1)}\right)\right]^{-1}$$

for any $x_1, x_2 \in M^2$ *and any* $0 < t_1 < t_2 < T$*, with strict inequality unless* $(M^2 \times [0, T), g)$ *is an expanding self-similar solution.*

Proof. Integrate the differential Harnack inequality along curves of the form $t \mapsto (t, \gamma(t))$. □

In fact, Theorem 11.8 is the trace version of the following more general "matrix Harnack inequality".

Theorem 11.10 (Matrix differential Harnack inequality[7]). *Along any Ricci flow $(M^2 \times [0,T), g)$ with positive curvature on a compact two-manifold M^2,*

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2 + \frac{1}{t}\mathrm{K}\right)|W|^2 - \nabla_W\nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}|U|^2 \geq 0 \tag{11.26}$$

for every time-dependent vector field W and two-form U. Moreover, if (11.26) *holds along a Ricci flow $(M^2 \times (0,T), g)$ on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless $(M^2 \times (0,T), g)$ is an expanding self-similar solution.*

Along any ancient Ricci flow $(M^2 \times (-\infty, T), g)$ with positive curvature on a compact two-manifold M^2,

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2\right)|W|^2 - \nabla_W\nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}|U|^2 \geq 0 \tag{11.27}$$

for every time-dependent vector field W and two-form U. Moreover, if (11.27) *holds along a Ricci flow $(M^2 \times (-\infty,T), g)$ on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless $(M^2 \times (-\infty,T), g)$ is a steady self-similar solution.*

[7] Richard S. Hamilton, "The Harnack estimate for the Ricci flow"

Proof. Motivated by various identities which hold on expanding (and steady) solitons, one considers the forms

$$Q(U,W) \doteqdot \left(\partial_t \mathrm{K} - \mathrm{K}^2\right) g(W,W) - \nabla_W\nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}g(U,U)$$

and

$$P(U,W) \doteqdot tQ(U,W) + \mathrm{K}g(W,W).$$

After some arduous computations (motivated by various identities which hold on solitons), it is possible to obtain a suitable differential inequality for P. □

11.4 The monotonicity formula for Perelman's functional

Given a Ricci flow $(M^2 \times I, g)$ on a compact surface M^2, define, for any $f : M^2 \times I \to \mathbb{R}$ and $\tau : I \to \mathbb{R}$, the functional

$$\mathscr{P}(f,g,\tau) \doteqdot \int_{M^2} \left[\tau\left(|\nabla f|^2 + 2\mathrm{K}\right) + f - 2\right](4\pi\tau)^{-1} \mathrm{e}^{-f} d\mu. \tag{11.28}$$

Observe that, when τ is identified with backwards time, Perelman's functional $\mathscr{P}$ is just a multiple of the functional F of (11.19) in the shrinking case, $\lambda > 0$ (with f replaced by $f - 2$).

Now, on a self-similarly shrinking Ricci flow $(M^2 \times (-\infty, 0), g)$ with potential function f and τ taken to be negative time,

$$\begin{aligned}\mathscr{P}(f, g, \tau) &= \int_{M^2} \left[\tau\left(|\nabla f|^2 + 2\mathrm{K}\right) + f - 2\right](4\pi\tau)^{-1}\mathrm{e}^{-f}d\mu \\ &= \frac{1}{2\pi}\int_{M^2}\left[\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda(f-2)\right)\mathrm{e}^{-f}d\mu \\ &= \frac{1}{\pi}\int_{M^2}\lambda f\mathrm{e}^{-f}d\mu\end{aligned}$$

due to (11.18) (and the choice of normalization of f). Since (by (11.21)) $\lambda\mathrm{e}^{-f}$ satisfies the conjugate heat equation, we find that

$$\begin{aligned}\frac{d}{dt}\mathscr{P}(f, g, \tau) &= \frac{1}{\pi}\int_{M^2}(\partial_t - \Delta)f\lambda\mathrm{e}^{-f}\,d\mu \\ &= \frac{1}{\pi}\int_{M^2}\left(\partial_t f - |\nabla f|^2\right)\lambda\mathrm{e}^{-f}\,d\mu \\ &= 0.\end{aligned}$$

The following remarkable *inequality* holds along a general Ricci flow on a compact surface.[8]

Theorem 11.11 (Perelman's monotonicity formula[9])**.** *Let $(M^2 \times I, g)$ be a Ricci flow on a compact surface M^2. If f and τ satisfy*

$$\begin{cases}(\partial_t + \Delta + 2\mathrm{K})f = |\nabla f|^2 + \frac{1}{\tau}, \\ \qquad\qquad \dfrac{d\tau}{dt} = -1,\end{cases}$$

then

$$\frac{d}{dt}\mathscr{P}(f, g, \tau) = 2\tau\int_{M^2}\left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{2\tau}g\right|^2\mathrm{e}^{-f}\,d\mu \tag{11.29}$$

so long as $\tau > 0$. In particular, the Perelman entropy

$$\mu(M^2, g_t, t_0 - t) \doteqdot \inf\left\{\mathscr{P}(g_t, f, t_0 - t) : \frac{1}{4\pi(t_0 - t)}\int_{M^2}\mathrm{e}^{-f}\,d\mu_t = 1\right\}$$

is nondecreasing for $t < t_0$ (strictly, unless (M^2, g_{t_0+t}) is a gradient shrinking soliton with potential $f(\cdot, t_0 + t)$).

[8] We omit the proof, as we will establish a generalization of the formula to all dimensions in §12.4.

[9] Perelman, "The entropy formula for the Ricci flow and its geometric applications"

11.5 Noncollapsing

Roughly speaking, a sequence of Riemannian surfaces (M_j^2, g_j) is said to collapse if some sequence of neighbourhoods $U_j \subset M_j^2$ and scales λ_j can be found such that $(U_j, \lambda_j g_j)$ resemble a one-dimensional manifold as $j \to \infty$. One precise way to quantify this is to ask for a sequence of points $p_j \in M_j$ such that

$$\mathrm{inj}_{g_j}(p_j)\sup_{B_{j\,\mathrm{inj}_{g_j}(p_j)}(p_j)}|\mathrm{K}|^{\frac{1}{2}} \leq j^{-1}, \tag{11.30}$$

where $\mathrm{inj}_g(p)$ denotes the INJECTIVITY RADIUS of (M^2, g) at p—the radius of the largest ball in (T_pM^2, g_p) on which the exponential map is a diffeomorphism.

Note that $\mathrm{inj}_g |\mathrm{K}|^{\frac{1}{2}}$ is scale invariant. Thus, if (11.30) holds, then, at the scale of the *curvature*, the injectivity radius degenerates to zero. On the other hand, at the scale of the *injectivity radius*, the curvature is tending towards zero in arbitrarily large regions, and at this scale the regions converge to a flat surface.

Example 19. Consider the constant sequence $(M_j^2, g_j) = (\mathbb{R}^2, g_{\mathrm{cigar}})$, where, in polar coordinates

$$g_{\mathrm{cigar}} = dr^2 + \tanh^2 r\, d\theta^2$$

is the cigar metric. If p_j are a sequence of points with $r_j \to \infty$, then, on the one hand, $\mathrm{inj}_j(p_j) \to \pi$ as $j \to \infty$. On the other hand, since $r_j \to \infty$ as $j \to \infty$, we may arrange, by passing to a subsequence, that $r_j - j\pi \to \infty$, and hence (recalling that $\mathrm{K} = 2\,\mathrm{sech}^2\, r$)

$$\begin{aligned}
\sup_{B_{j\,\mathrm{inj}_j(p_j)}(p_j)} \mathrm{K} &\leq \sup_{B_{j\pi}(p_j)} \mathrm{K} \\
&\leq \sup_{r_j - j\pi \leq r \leq r_j + j\pi} \mathrm{K} \\
&= \mathrm{K}(r_j + j\pi)) \\
&= 2\,\mathrm{sech}^2(r_j - j\pi) \\
&= o(1) \quad \text{as} \quad j \to \infty.
\end{aligned}$$

So the sequence is collapsing. ■

11.5.1 *The isoperimetric estimate*

The RELATIVE ISOPERIMETRIC CONSTANT of a Riemannian two-sphere $(M^2 \cong S^2, g)$ is defined to be

$$\mathcal{I}(M^2, g) \doteqdot \inf_{\Gamma} \mathrm{relength}(\Gamma, g),$$

where the infimum is taken over all SEPARATING CURVES—regular Jordan curves $\Gamma \subset M^2$ which[10] separate M^2 into two topological disks, Ω_1 and Ω_2—and the RELATIVE LENGTH of a separating curve is defined by

$$\mathrm{relength}(\Gamma, g) \doteqdot \frac{\mathrm{length}(\Gamma, g)}{\mathrm{length}(\overline{\Gamma}, \overline{g})},$$

where the COMPARISON ARC $\overline{\Gamma}$ is the (unique up to isometry) shortest Jordan curve which separates the round sphere $(S^2, \overline{g})$ of the same area as (M^2, g) into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ of the same areas as Ω_1 and Ω_2, respectively.

[10] Necessarily, by the Schoenflies theorem.

Obviously, the relative isoperimetric constant of a round sphere is one. Moreover, since

$$\mathrm{relength}(\Gamma, g) \to 1 \text{ as } \mathrm{length}(\Gamma, g) \to 0,$$

the relative isoperimetric constant cannot exceed one on any two-sphere (M^2, g). In fact, $\mathcal{I}(M^2, g) < 1$ unless (M^2, g) is isometric to a round sphere.

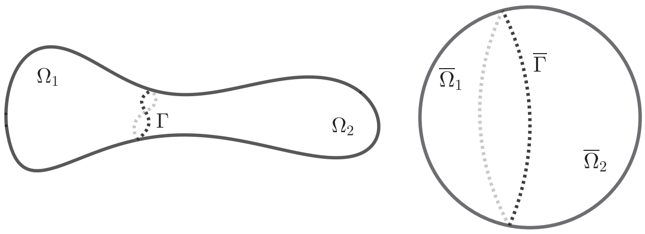

Figure 11.1: Given a curve, Γ, separating a surface $(M^2 \cong S^2, g)$, into regions Ω_1 and Ω_2, the comparison curve, $\overline{\Gamma}$, is the shortest curve separating M^2 into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ which when measured in the round geometry on M^2 of the same area as g have the same areas as Ω_1 and Ω_2, respectively, as measured in the original geometry.

Hamilton proved that the isoperimetric constant of a Riemannian sphere does not decrease under Ricci flow.

Proposition 11.12. *Let $(M^2 \times [0, T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\frac{d}{dt}\mathcal{I}(M^2, g_t) \geq 0$$

in the viscosity sense[11] *whenever $\mathcal{I}(M^2, g_t) < 1$. In particular,*

$$\mathcal{I}(M^2, g_t) \geq \mathcal{I}(M^2, g_0).$$

[11] This is a weak formulation of the differential inequality $\frac{du}{dt} \geq 0$ which applies to any continuous function. It asserts, for every $t_0 \in (0, T)$, that every smooth function $\varphi : [0, T) \to \mathbb{R}$ which touches u from below at t_0, in the sense that $u \leq \varphi$ for t in a backward neighbourhood $(t_0 - \delta, t_0]$ of t_0 with equality at t_0, satisfies $\frac{d\varphi}{dt}(t_0) \geq 0$.

Sketch of the proof. First note that, given any separating curve Γ for a surface (M^2, g), the first variation formula for the length of a separating curve in the comparison surface $(M^2, \overline{g})$, subject to the area constraint, guarantees that any comparison curve $\overline{\Gamma}$ has constant curvature. For such curves, we have the formula

$$\frac{4\pi}{\mathrm{length}^2(\overline{\Gamma}, \overline{g})} = \frac{1}{\mathrm{area}(\overline{\Omega}_1, \overline{g})} + \frac{1}{\mathrm{area}(\overline{\Omega}_2, \overline{g})},$$

where $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are the two regions bounded by Γ, which gives the formula

$$\mathrm{relength}(\Gamma, g) = \frac{\mathrm{length}(\Gamma, g)}{\sqrt{4\pi}} \left(\frac{1}{\mathrm{area}(\Omega_1, g)} + \frac{1}{\mathrm{area}(\Omega_2, g)} \right)^{\frac{1}{2}},$$

where Ω_1 and Ω_2 are the regions bounded by Γ.

Recall now that the length and area of a variation $\{\Gamma_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ of $\Gamma = \Gamma_0$ vary according to

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{length}(\Gamma_\varepsilon) = -\int_\Gamma g(\vec{\kappa}, V)\, ds,$$

and

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \operatorname{area}(\Omega_\varepsilon) = \int_\Gamma g(\mathsf{N}, V)\, ds\,,$$

where V is the variation field and N is the outward unit normal corresponding to the choice of bounded region, Ω. It follows that

$$\frac{\int_\Gamma \kappa g(\mathsf{N}, V)\, ds}{\int_\Gamma g(\mathsf{N}, V)\, ds} = \frac{1}{2} \operatorname{length}(\Gamma, g) \left(\frac{1}{\operatorname{area}(\Omega, g)} - \frac{1}{\operatorname{area}(M^2 \setminus \Omega, g)} \right)$$

at a minimizer of the relative length for any (nontrivial) variation V. From this we may conclude that a miminizer has constant curvature,

$$\kappa \equiv \frac{1}{2} \operatorname{length}(\Gamma, g) \left(\frac{1}{\operatorname{area}(\Omega, g)} - \frac{1}{\operatorname{area}(M^2 \setminus \Omega, g)} \right). \tag{11.31}$$

Consider now the constant distance variation, $\Gamma_\varepsilon = \{\exp_p \varepsilon \mathsf{N}_p : p \in \Gamma\}$. The second variation identities, along this variation, are given by

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \operatorname{length}(\Gamma_\varepsilon) = -\int_\Gamma \mathrm{K}\, ds$$

and

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \operatorname{area}(\Omega_\varepsilon) = \int_\Gamma \kappa\, ds\,.$$

(The first of these is established by differentiating the Gauss–Bonnet formula, $\int_{\Omega_\varepsilon} \mathrm{K}\, d\mu + \int_{\Gamma_\varepsilon} \kappa_\varepsilon\, ds = 2\pi$ while the second follows from the identity $\frac{d}{d\varepsilon} \operatorname{area}(\Omega_\varepsilon, g) = \operatorname{length}(\Gamma_\varepsilon, g)$.) Combining these and recalling (11.31), we conclude that

$$\int_\Gamma \mathrm{K}\, ds \leq \frac{\operatorname{length}^3(\Gamma, g)}{\operatorname{area}(\Omega, g) \operatorname{area}(M^2 \setminus \Omega, g)} \tag{11.32}$$

at a minimizer Γ of the relative length.

Now, if $\mathcal{I}(M^2, g) < 1$, then (since $\operatorname{relength}(\Gamma, g)$ approaches 1 as $\operatorname{length}(\Gamma, g)$ approaches 0) a minimizing sequence[12] of separating curves Γ_j will have lengths bounded uniformly from below. It is then possible to extract a suitable *weak* limit curve, Γ. Though this limiting curve may not be smooth *a priori*, the vanishing of the first variation of the relative length at Γ ensures that Γ has constant curvature in the corresponding weak sense, which guarantees that it is smooth (and connected, else a better constant is given by one of the components).

[12] I.e. $\operatorname{relength}(\Gamma_j, g) \to \mathcal{I}(M^2, g)$ as $j \to \infty$.

Now suppose that the metrics $\{g_t\}_{t\in[0,T)}$ on M^2 evolve by Ricci flow. Given $t_0 \in (0, T)$, if $\mathcal{I}(M^2, g_{t_0}) < 1$, then we can find some minimizing curve, Γ_{t_0}, as above. Given any variation $\{\Gamma_t\}_{t\in(t_0-\delta,t_0]}$ of Γ_{t_0}, the inequality $\operatorname{relength}(\Gamma_t, g_t) \geq \mathcal{I}(M^2, g_t)$ holds for $t \in (t_0 - \delta, t_0]$, with equality at time t_0. Thus, if φ is a lower support for $\mathcal{I}(M^2, g_t)$ at time t_0, then $\varphi(t) \leq \operatorname{relength}(\Gamma_t, g_t)$ with equality at time t_0, and hence, at time t_0,

$$\frac{d}{dt}\varphi \geq \frac{d}{dt} \operatorname{relength}(\Gamma_t, g_t)\,.$$

If we take $\{\Gamma_t\}_{t\in(t_0-\delta,t_0]}$ to be the constant distance variation (in the outwards direction with respect to a choice of bounded domain Ω), then

$$\frac{d}{dt}\operatorname{length}(\Gamma_t, g_t) = \int_{\Gamma_t}(\kappa - \mathrm{K})\,ds\,,$$

$$\begin{aligned}\frac{d}{dt}\operatorname{area}(\Omega_t, g_t) &= \operatorname{length}(\Gamma_t, g_t) - 2\int_{\Omega_t}\mathrm{K}\,d\mu\\ &= \operatorname{length}(\Gamma_t, g_t) + 2\int_{\Gamma_t}\kappa\,d\mu - 4\pi\end{aligned}$$

and

$$\frac{d}{dt}\operatorname{area}(M^2\setminus\Omega_t, g_t) = -\operatorname{length}(\Gamma_t, g_t) - 2\int_{\Gamma_t}\kappa\,d\mu - 4\pi\,,$$

where $\Gamma_t = \partial\Omega_t$ is either choice of orientation. Thus,

$$\begin{aligned}&\frac{\frac{d}{dt}\operatorname{relength}(\Gamma_t, g_t)}{\operatorname{relength}(\Gamma, g_t)}\\ &= \frac{\int_{\Gamma_t}(\kappa-\mathrm{K})\,ds}{\operatorname{length}(\Gamma, g_t)}\\ &\quad+\frac{2\pi\left(\operatorname{area}^2(\Omega_t,g_t)+\operatorname{area}^2(M^2\setminus\Omega_t,g_t)\right)}{\operatorname{area}(\Omega_t,g_t)\operatorname{area}(M^2\setminus\Omega_t,g_t)\left(\operatorname{area}(\Omega_t,g_t)+\operatorname{area}(M^2\setminus\Omega_t,g_t)\right)}\\ &\quad-\frac{1}{2}\left(\operatorname{length}(\Gamma_t,g_t)+2\int_{\Gamma_t}\kappa\,d\mu\right)\left(\frac{1}{\operatorname{area}(\Omega_t,g_t)}-\frac{1}{\operatorname{area}(M^2\setminus\Omega_t,g_t)}\right).\end{aligned}$$

Recalling (11.31) and (11.32), we find, at time $t = t_0$, that

$$\begin{aligned}\frac{d}{dt}\ln\varphi &\geq \frac{d}{dt}\ln\operatorname{relength}(\Gamma_t, g_t)\\ &\geq \frac{2\pi\left[\operatorname{area}^2(\Omega_t,g_t)+\operatorname{area}^2(M^2\setminus\Omega_t,g_t)\right](1-\operatorname{relength}^2(\Gamma_t,g_t))}{\operatorname{area}(\Omega_t,g_t)\operatorname{area}(M^2\setminus\Omega_t,g_t)\left[\operatorname{area}(\Omega_t,g_t)+\operatorname{area}(M^2\setminus\Omega_t,g_t)\right]}.\end{aligned}$$

The first claim follows.

To prove the second claim, it suffices to establish that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \in [0, T)$ for any $\varepsilon > 0$. Note that the inequality holds strictly at time $t = 0$ for any positive ε. Suppose then that some $\varepsilon > 0$ and $t_0 \in (0, T)$ can be found such that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \leq t_0$, but with equality at time $t = t_0$. But then the function

$$\varphi(t) \doteqdot \mathcal{I}(\Gamma_0) - \varepsilon(1+t)$$

is a lower support for $\mathcal{I}$ at time $t = t_0$, and hence

$$0 \leq \frac{d}{dt}\varphi = -\varepsilon < 0,$$

which is absurd. □

Combining this with Klingenberg's lemma yields the following lower bound for the injectivity radius.

Corollary 11.13. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\mathrm{inj}^2(M^2, g_t) \geq \frac{\pi}{4}\frac{\mathcal{I}(M^2, g_0)}{\mathrm{K}_{\max}(t)}. \tag{11.33}$$

11.5.2 A lower bound for area at the scale of the curvature

Recall, from Theorem 11.11, that, for any choice of backwards time $\tau(t) = t_0 - t$, Perelman's entropy

$$\mu(M^2, g_t, \tau(t)) = \inf\left\{\mathscr{P}(g_t, f, \tau(t)) : \frac{1}{4\pi\tau(t)}\int_{M^2} \mathrm{e}^{-f}\, d\mu_t = 1\right\}$$

is nondecreasing along a Ricci flow $(M^2 \times [0,T), g)$ whilever $\tau(t) > 0$. Given $t_0 \in [0,T)$, set $\tau = t_0 + r^2 - t$ and consider the test function $u(\cdot, t_0) = (4\pi r^2)^{-\frac{n}{2}}\mathrm{e}^{-f(\cdot,t_0)}$ with $\mathrm{e}^{-f(\cdot,t_0)} = A\chi_{B_r(x_0,t_0)}$. Observe that, in order to satisfy the constraint

$$\int_{M^2} u(\cdot, t_0) d\mu_{t_0} = 1,$$

we should take $A \sim \frac{\mathrm{area}(B_r(x_0,t_0),t_0)}{r^2}$. Monotonicity of the entropy then implies

$$\begin{aligned}
\mu(M^2, g_0, t_0 + r^2) &\leq \mu(M^2, g_{t_0}, r^2)\\
&\leq \mathscr{P}(g, f(\cdot, t_0), r^2)\\
&= \int_{M^2}\left[r^2\left(|\nabla f|^2 + 2\mathrm{K}\right) + f - 2\right](4\pi r^2)^{-1}\mathrm{e}^{-f} d\mu\\
&\lesssim r^2 \max_{B_r(x_0,t_0)} \mathrm{K}(\cdot, t_0) + \ln\frac{\mathrm{area}(B_r(x_0,t_0),t_0)}{r^2}.
\end{aligned}$$

Thus, if $\mathrm{K}(\cdot, t_0) \lesssim r^{-2}$, then we obtain the lower area bound

$$\frac{\mathrm{area}(B_r(x_0,t_0),t_0)}{r^n} \geq \kappa(M^2, g_0, T).$$

I.e. areas are bounded uniformly from below at the scale of the curvature. By Proposition 9.20, this yields a uniform lower bound for the injectivity radius at the scale of the curvature, so the flow is noncollapsing.

Note though, that this argument is not quite rigorous, as the test function is not smooth (we took the gradient term to be zero), but it can easily be made so by introducing a cut-off function.[13]

[13] We omit the proof as we shall carry it out in general dimensions in §12.4.

Theorem 11.14. [14] *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a compact surface M^2. Given $(x,t) \in M^2 \times [0,T)$, if $|\mathrm{K}| \leq r^{-2}$ on $B_r(x,t)$, $r \leq 1$, then*

$$\mathrm{area}(B_r(x,t),t) \geq \kappa r^2,$$

where $\kappa = \kappa(M^2, g_0, T)$.

[14] Perelman, "The entropy formula for the Ricci flow and its geometric applications".

11.6 Uniformization of surfaces by Ricci flow

Recall the lower curvature bound

$$\mathrm{K} - \kappa \gtrsim \begin{cases} -\dfrac{1}{t^2} \text{ as } t \to \infty \text{ if } \chi(M^2) < 0 \\ -\dfrac{1}{t} \text{ as } t \to \infty \text{ if } \chi(M^2) = 0 \\ -\dfrac{1}{1-\chi(M^2)t} \text{ as } t \to \dfrac{1}{\chi(M^2)} \text{ if } \chi(M^2) > 0. \end{cases}$$

from (11.8). We shall obtain a complimentary upper bound by seeking an estimate which is saturated by soliton solutions. Recall that, on a gradient Ricci soliton, the potential function f satisfies

$$-\Delta f = 2(\mathrm{K} - \kappa). \tag{11.34}$$

On the other hand, since its right hand side has zero average, the equation (11.34) admits a solution f on *any* compact two-dimensional Ricci flow. Moreover, by the maximum principle, the solution f is unique up to the addition of a function of time.

Lemma 11.15. *Every Ricci flow $(M^2 \times [0, T), g)$ on a compact two-manifold M^2 admits a curvature potential function satisfying*

$$(\partial_t - \Delta)f = 2\kappa f$$

and hence, assuming $\mathrm{area}(M^2, 0) = 4\pi$,

$$\frac{\min_{M^2\times\{0\}} f}{1-\chi(M^2)t} \le f \le \frac{\max_{M^2\times\{0\}} f}{1-\chi(M^2)t}. \tag{11.35}$$

Proof. Since, for any function u,

$$\partial_t \Delta u = \Delta \partial_t u + 2\mathrm{K}\Delta u,$$

we find that

$$\begin{aligned} \Delta\partial_t f &= \partial_t \Delta f - 2\mathrm{K}\Delta f \\ &= -2\partial_t(\mathrm{K}-\kappa) + 4\mathrm{K}(\mathrm{K}-\kappa) \\ &= -2\Delta(\mathrm{K}-\kappa) - 4(\mathrm{K}^2-\kappa^2) + 4\mathrm{K}(\mathrm{K}-\kappa) \\ &= \Delta\Delta f + 2\kappa\Delta f \\ &= \Delta(\Delta f + 2\kappa f). \end{aligned}$$

That is,

$$\Delta(\partial_t f - \Delta f - 2\kappa f) = 0.$$

So $\partial_t f - \Delta f - 2\kappa f$ is a function of t only. By exploiting the freedom to add a function of t to f, we can easily guarantee that

$$(\partial_t - \Delta)f - 2\kappa f = 0$$

as claimed. The second claim then follows from the maximum principle, since, under the area normalization, $\kappa = \frac{\chi(M^2)}{1-\chi(M^2)t}$ □

Recall from (11.18) that, on a two-dimensional Ricci soliton,

$$0 = \nabla \left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \kappa f \right).$$

That is, $\mathrm{K} + \frac{1}{2}|\nabla f|^2 - \kappa f$ is a function of time only. Consider then, on a general (compact) two dimensional Ricci flow, the function

$$F \doteqdot \mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \kappa f$$

where f is a curvature potential satisfying Lemma 11.15.

Proposition 11.16. *The function F satisfies*

$$(\partial_t - \Delta)F = 2\kappa F - 2\left|\nabla^2 f - \tfrac{1}{2}\Delta f g\right|^2 \tag{11.36}$$

and hence

$$F \leq \frac{\max_{M^2 \times \{0\}} F}{1 - \chi(M^2)t} \tag{11.37}$$

with strict inequality unless $(M^2 \times I, g)$ is a soliton.

Proof. We leave the verification of (11.36) as an exercise. The inequality (11.37) follows from the maximum principle, with strict inequality unless it holds identically. But in that case (11.36) implies that $\nabla^2 f - \frac{1}{2}\Delta f g = 0$. The final claim follows. □

This is an extremely useful estimate. For instance, we immediately obtain precise control on the maximal time of existence.

Corollary 11.17. *Let $(M^2 \times [0, T), g)$ be the maximal Ricci flow of a compact Riemannian surface (M^2, g_0). If $\chi(M^2) \leq 0$, then $T = \infty$. If $\chi(M^2) > 0$, then $T = \frac{1}{\chi(M^2)}$.*

Proof. By (11.35) and (11.37), there is a constant $C < \infty$ such that

$$\mathrm{K} \leq \frac{C}{1 - \chi(M^2)t} \left(1 - \frac{\chi(M^2)}{1 - \chi(M^2)t} \right). \tag{11.38}$$

So the claim follows from the long-time existence theorem (Theorem 9.16). □

In fact, the estimate (11.37) in conjunction with the lower bound (11.8) will be sufficient to establish infinite time existence and convergence of the flow in case $\chi(M^2) \leq 0$. The case $\chi(M^2) > 0$ is somewhat trickier due to the finite time singularity. In that case, we analyze the singularity by rescaling and applying Theorem 9.19. The rescaling normalizes the curvature, but we still need to establish lower bounds for the injectivity radius. Note that, in the elliptic case, $\chi(M^2) > 0$, the universal cover is S^2 (which is compact); so it suffices to work on S^2, in which case Corollary 11.13 yields the desired bound.

Theorem 11.18 (Chow and Hamilton[15]). *Given a compact Riemannian surface (M^2, g_0), let $(M \times [0, T), g)$ be the maximal Ricci flow starting at (M^2, g_0).*

- *If $\chi(M^2) > 0$, then $T < \infty$ and $\frac{1}{2(T-t)} g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = +1$ as $t \to T$.*
- *If $\chi(M^2) = 0$, then $T = \infty$ and g_t converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = 0$ as $t \to T$.*
- *If $\chi(M^2) < 0$, then $T = \infty$ and $\frac{1}{2t} g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = -1$ as $t \to \infty$.*

[15] Chow, "The Ricci flow on the 2-sphere"; Richard S. Hamilton, "The Ricci flow on surfaces"

Sketch of the proof. Consider first the case $\chi(M^2) = 0$. In this case, $\kappa = 0$, and (11.8) becomes

$$\mathrm{K} \geq -\frac{1}{2t}.$$

The uniform upper bound for K of (11.38) then implies a uniform bound for $\nabla \mathrm{K}$ via the Bernstein estimates. Since the average of K is zero, we are then able to conclude that $\mathrm{K} \to 0$ as $t \to \infty$. Convergence of g_t to a limit metric then follows from the Ricci flow equation via the identity

$$-\frac{d}{dt} \log g_{(x,t)}(v, v) = 2\mathrm{K}(x, t) \tag{11.39}$$

for any $x \in M^2$ and any $v \in T_x M^2$. The limit metric is flat and the convergence is smooth, since the higher order Bernstein estimates and the interpolation inequality yield $\mathrm{K} \to 0$ to *all* orders as $t \to \infty$.

The hyperbolic case, $\chi(M^2) < 0$, may be treated similarly as the flat case, $\chi(M^2) = 0$. We omit the details.

The elliptic case, $\chi(M^2) > 0$, is more difficult. But at least we may work on the universal cover, S^2 (since it is compact). The lower bound (11.33) for the injectivity radius allows us to blow-up at the final time, $T < \infty$, to obtain an ancient limit Ricci flow. Note that (by the ODE comparison principle) $\max_{M^2 \times \{t\}} \mathrm{K} \geq \frac{1}{2(T-t)}$. Assume first that $\max_{M^2 \times \{t\}} \mathrm{K} \leq C(T-t)^{-1}$ (the expected rate of blow-up). Choose any sequence of times $t_j \nearrow T$ and points $x_j \in M^2$ such that

$$r_j^{-2} \doteqdot \max_{M^2 \times [0, t_j]} \mathrm{K} = \mathrm{K}_{(x_j, t_j)}$$

and consider the pointed rescaled Ricci flows $(M^2 \times I_j, x_j, g_j)$, where $I_j \doteqdot [-r_j^{-2} t_j, r_j^{-2}(T - t_j))$ and $(g_j)_{(x,t)} \doteqdot r_j^{-2} g(x, r_j^2 t + t_j)$. Observe that the curvature K_j of the rescaled Ricci flow satisfies

$$\mathrm{K}_j(x, t) = r_j^2 \mathrm{K}(x, r_j^2 t + t_j) \leq \frac{C r_j^2}{T - t_j - r_j^2 t} = \frac{C}{r_j^{-2}(T - t_j) - t} \leq \frac{2C}{1 - 2t}.$$

Since, by Corollary 11.13,

$$\mathrm{inj}(M^2,(g_j)_t) \geq \frac{\sqrt{\pi\,\mathcal{I}(M^2,g_0)}}{2},$$

some subsequence of the pointed rescaled Ricci flows $(M^2\times I_j, x_j, g_j)$ converges locally uniformly in the smooth sense to a limit ancient Ricci flow $(M^2_\infty\times(-\infty,1), g_\infty)$. Since $\mathrm{area}(M^2,g_t)\to 0$ as $t\to T$, Proposition 11.12 implies that $\mathrm{diam}(M^2,g_t)\to 0$ as well. Proposition 9.7 then implies that

$$\begin{aligned}\mathrm{diam}(M^2,(g_j)_t) &= r_j^{-1}\,\mathrm{diam}(M^2, g_{r_j^2t+t_j})\\ &\leq 10r_j^{-2}(T-t_j-r_j^2t)\\ &\leq C(1-2t).\end{aligned}$$

So the limit is compact, and hence $M^2_\infty = M^2\cong S^2$.

Next, we claim that $\max_{M^2\times\{t\}} F/\kappa$ is constant on the limit flow. Recall that $\max_{M^2\times\{t\}} F/\kappa$ is nonincreasing on the original flow since

$$(\partial_t-\Delta)\frac{F}{\kappa} = -2\left|\nabla^2 f-\tfrac{1}{2}\Delta f g\right|^2.$$

In particular, $\max_{M^2\times\{t\}} F/\kappa$ takes a limit as $t\to T$. Now, since both numerator and denominator scale like curvature, we have, for any $a<b\in(-\infty,1)$,

$$\max_{M^2\times\{b\}}\frac{F_j}{\kappa_j}-\max_{M^2\times\{a\}}\frac{F_j}{\kappa_j} = \max_{M^2\times\{r_j^2b+t_j\}}\frac{F}{\kappa}-\max_{M^2\times\{r_j^2a+t_j\}}\frac{F}{\kappa}$$

for all j sufficiently large. But both $r_j^2a+t_j$ and $r_j^2a+t_j$ tend to T, so the right hand side tends to zero. So $\max_{M^2\times\{t\}} F/\kappa$ is indeed constant on the limit flow. But then $\frac{F}{\kappa}$ must be constant, due to the strong maximum principle. We conclude that

$$\nabla^2 f-\tfrac{1}{2}\Delta f g\equiv 0$$

on the limit flow, which must therefore be a gradient Ricci soliton[16], and hence the shrinking sphere by Theorem 11.3. The theorem now follows from bootstrapping arguments.

[16] Alternatively, we could have invoked the Chow–Hamilton entropy and Proposition 11.2 here, since non-flat compact ancient Ricci flows on surfaces have positive curvature (by Corollary 9.12).

It remains to prove that $\mathrm{K}(T-t)$ remains bounded. Suppose then that, to the contrary,

$$\limsup_{t\nearrow T}\max_{M^2\times\{t\}}\mathrm{K}(T-t)=\infty.$$

For each j, choose $(x_j,t_j)\in M^2\times[0,T)$ so that

$$(T-j^{-1}-t_j)\mathrm{K}(x_j,t_j) = \max_{M^2\times[0,T-j^{-1}]}(T-j^{-1}-t)\mathrm{K}$$

and set $r_j^{-2} \doteqdot \mathrm{K}(x_j, t_j)$. Consider the pointed rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$, where $\alpha_j \doteqdot -r_j^{-2} t_j$, $\omega_j \doteqdot r_j^{-2}(T - j^{-1} - t_j)$ and $(g_j)_{(x,t)} \doteqdot r_j^{-2} g_{(x, r_j^2 t + t_j)}$. Observe in this case that

$$\alpha_j \to -\infty, \;\; \omega_j \to \infty,$$

and

$$\mathrm{K}_j(x,t) = r_j^2 \mathrm{K}(x, r_j^2 t + t_j) \leq \frac{T - j^{-1} - t_j}{T - j^{-1} - r_j^2 t + t_j} = \frac{\omega_j}{\omega_j - t},$$

which is uniformly bounded on any compact time interval for j sufficiently large. Since, by Proposition 11.13, the injectivity radii remain uniformly bounded from below after rescaling, some subsequence of the pointed, rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$ must converge to an eternal limit pointed Ricci flow $(M^2_\infty \times (-\infty, \infty), x_\infty, g_\infty)$. Since this Ricci flow is the limit of compact Ricci flows, it satisfies the differential Harnack inequality. But, by construction,

$$\mathrm{K} \leq \limsup_{j \to \infty} \frac{\omega_j}{\omega_j - t} = 1 = \mathrm{K}(x_\infty, 0).$$

Thus, at $(x_\infty, 0)$, $\partial_t \mathrm{K} = 0$ and $\nabla \mathrm{K} = 0$, and hence the rigidity case of the differential Harnack inequality implies that $(M^2_\infty \times (-\infty, \infty), g_\infty)$ is a steady soliton, which must be a cigar by Theorem 11.4 and the curvature normalization at $(x_\infty, 0)$. But the cigar violates the (scale invariant) lower bound for the isoperimetric constant (which passes to the limit as it is scale invariant and lower semi-continuous under local uniform convergence). This completes the proof. $\square$

The original argument of Hamilton and Chow made use of the Kazdan–Warner identity—which relies on the uniformization theorem—to establish Theorem 11.3. The argument presented here for Theorem 11.3 (which does not require the uniformization theorem) was pointed out by Chen–Lu–Tian.[17]

A different proof of Theorem 11.18 was later found by Andrews–Bryan[18] and Bryan[19] (following Hamilton[20]). They were able to obtain a very sharp estimate for the isoperimetric profile under Ricci flow, sharp enough indeed to obtain sharp control on the curvature (which appears in the second variation of the isoperimetric profile), and thereby obtain convergence directly.

[17] X. Chen, Lu, and Tian, "A note on uniformization of Riemann surfaces by Ricci flow".

[18] Andrews and Bryan, "Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere".

[19] Bryan, "Curvature bounds via an isoperimetric comparison for Ricci flow on surfaces".

[20] Richard S. Hamilton, "An isoperimetric estimate for the Ricci flow on the two-sphere".

11.7 *Exercises*

Exercise 11.1. Suppose that the two metrics g and g_0 on a surface M^2 are related by $g = \mathrm{e}^{-2u} g_0$ for some function u. Show that the respective sectional curvatures K and K_0 are related by

$$\mathrm{K} = e^{2u}(\Delta_0 u + \mathrm{K}_0),$$

where Δ_0 is the Laplace–Beltrami operator induced by g_0.

Exercise 11.2. Let (M^2, g, f) be a two-dimensional gradient Ricci soliton. Show that

$$K \doteqdot \mathrm{J}(\nabla f)$$

is a Killing vector field, where $\mathrm{J} : TM^2 \to TM^2$ denotes counterclockwise rotation in the fibres through 90 degrees. HINT: *first show that* J *is parallel*.

Exercise 11.3. Show that a solution to the heat equation $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ satisfies

$$\nabla^2 \log u + \frac{n}{2t} = 0.$$

if and only if it is a fundamental solution.

Exercise 11.4. Prove that

$$\Delta \log u + \frac{1}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation. HINT: Consider the function $P \doteqdot 2t\Delta \log u + 1$.

Exercise 11.5. Prove that

$$\nabla^2 \log u + \frac{\mathrm{I}}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation, where I is the Euclidean inner product. HINT: Consider the function $P \doteqdot 2t\nabla_V \nabla_V \log u + I$ for any fixed vector $V \in S^n$.

Exercise 11.6. Set $U = V \wedge W$ in (11.26) and trace with respect to W, and then optimize with respect to V to obtain (11.24).

Exercise 11.7. ANDREWS' INEQUALITY[21] states that

$$\tfrac{n}{n-1} \int_{M^n} \varphi^2 \, d\mu \leq \int_{M^n} |F|^2 + \int_{M^n} \mathrm{Rc}^{-1} \left(\nabla \varphi - \operatorname{div} F, \nabla \varphi - \operatorname{div} F \right) d\mu$$

on any compact Riemannian manifold (M^n, g) with positive Ricci curvature for every zero-average smooth function φ and every trace-free, symmetric, smooth two-tensor F, with equality only if

$$-\tfrac{n-1}{n} F = \nabla^2 f - \tfrac{1}{n} \Delta f g \quad \text{and} \quad \tfrac{n-1}{n} \operatorname{div} F = \tfrac{n-1}{n} \nabla \varphi - \mathrm{Rc}(\nabla f)$$

for some (any) potential function f for φ (solution to $-\Delta f = \varphi$).

[21] See Chow, Lu, and Ni, *Hamilton's Ricci flow*, Theorem B.18 for a proof.

1. Using Andrews' inequality, show that, on any compact Riemannian manifold (M^n, g) with positive Ricci curvature,

$$\tfrac{n}{n-1}\int_{M^n}(\mathrm{R}-\rho)^2 d\mu \le \alpha^2 \int_{M^n} \left|\mathring{\mathrm{Rc}}\right|^2 d\mu + \left(1-\alpha\left(\tfrac{1}{2}-\tfrac{1}{n}\right)\right)^2 \int_{M^n} \mathrm{Rc}^{-1}(\nabla \mathrm{R}, \nabla \mathrm{R})\, d\mu$$

for any $\alpha \in \mathbb{R}$, with equality only if

$$-\tfrac{n-1}{n}\alpha\mathring{\mathrm{Rc}} = \nabla^2 f - \tfrac{1}{n}\Delta f g \quad\text{and}\quad \left(\tfrac{1}{2}+\tfrac{1}{n}\right)\nabla \mathrm{R} = \tfrac{n}{n-1}\,\mathrm{Rc}(\nabla f)$$

for some (any) scalar curvature potential function f (solution to $-\Delta f = \mathrm{R} - \rho$), where ρ denotes the average scalar curvature.

2. (HAMILTON'S INEQUALITY) Deduce (or otherwise prove) that, on any compact Riemannian surface (M^2, g) with positive curvature,

$$\int_{M^2} (\mathrm{K}-\kappa)^2 d\mu \le \frac{1}{2}\int_{M^2} \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}} d\mu \qquad (11.40)$$

with equality only if

$$\nabla^2 f - \tfrac{1}{n}\Delta f g = 0 \quad\text{and}\quad \nabla \mathrm{K} = \mathrm{K}\,\nabla f$$

for some (any) curvature potential function f (solution to $-\Delta f = 2(\mathrm{K}-\kappa)$).

Exercise 11.8. Use Hamilton's inequality to establish the monotonicity formula for the Chow–Hamilton entropy (Proposition 11.2).