ANCIENT SOLUTIONS TO MEAN CURVATURE FLOW

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Abstract. We describe our recent construction and classification of rotationally symmetric ancient 'pancake' solutions to mean curvature flow. The full details can be found in [5].

1. Introduction

A smooth one-parameter family $\{M_t^n\}_{t\in I}$ of smoothly immersed hypersurfaces M_t^n of $\mathbb{R}^{n+1}$ is a (classical) *solution to mean curvature flow* if there exists a smooth one parameter family $X : M^n \times I \to \mathbb{R}^{n+1}$ of immersions $X(\cdot,t) : M^n \to \mathbb{R}^{n+1}$ with $M_t^n = X(M^n,t)$ satisfying

$$\frac{\partial X}{\partial t}(x,t) = \vec{H}(x,t) \quad \text{for all} \quad (x,t) \in M^n \times I\,,$$

where $\vec{H}(\cdot,t)$ is the mean curvature vector field of $X(\cdot,t)$. The mean curvature flow is the L^2-gradient flow of the area functional and hence, roughly speaking, deforms a hypersurface in such a way as to decrease its area most rapidly.

An *ancient* solution to a geometric flow, such as mean curvature flow, is one which is defined on a time interval of the form $I = (-\infty, T)$, where $T \leq \infty$. Ancient solutions are of interest due to their natural role in the study of high curvature regions of the flow; namely, they arise as blow-up limits at singularities [10, 14, 15, 16, 17, 23, 24]. A special class of ancient solutions are the *translating solutions*. As the name suggests, these are solutions $\{M_t^n\}_{t\in(-\infty,\infty)}$ which evolve by translation: $M_{t+s}^n = M_t^n + se$ for some fixed vector $e \in \mathbb{R}^{n+1}$.

Ancient solutions to mean curvature flow are also relevant in conformal field theory, where, "to lowest order in perturbation theory, they describe the ultraviolet regime of the boundary renormalization group equation of Dirichlet sigma models" [4].

Further interest in ancient and translating solutions to geometric flows arise from their rigidity properties, which are analogous to those of complete minimal surfaces, harmonic maps and Einstein metrics; for example, when $n \geq 2$, under certain geometric conditions — uniform convexity, bounded eccentricity, type-I curvature decay or bounded isoperimetric ratio, for instance — the only compact[1], convex (or noncollapsing) ancient solutions to mean curvature flow are shrinking spheres [18, 11]. In fact, the convexity/noncollapsing condition can be weakened to a uniform bound for $|A|^2/H^2$, where A is the second fundamental form of the solution [19].

Compact ancient solutions to mean curvature flow are closely related to translating solutions. Consider a *complete* solution $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, to the PDE

$$\operatorname{div}\left(\frac{Du}{\sqrt{\sigma^2+|Du|^2}}\right) = -\frac{1}{\sqrt{\sigma^2+|Du|^2}}\,. \tag{1}$$

When $\sigma = 0$, u is the *arrival time* of a mean convex ancient solution to mean curvature flow in $\mathbb{R}^n$ (i.e. the level sets of u form a mean convex ancient solution). When $\sigma = 1$, the graph of u is the time zero slice of a mean convex translating solution to mean curvature flow in $\mathbb{R}^{n+1}$ with bulk velocity $-e_{n+1}$. The converse is also true: Every compact, mean convex ancient solution to mean curvature flow (resp. mean convex, proper translating solution) gives rise to a complete solution to (1) with $\sigma = 0$ (resp. $\sigma = 1$). This formal connection was exploited by X.-J. Wang, who proved a series of remarkable results for convex translating solutions and compact, convex ancient solutions to mean curvature flow [22].

2. Ancient ovaloids vs ancient pancakes

Even under strong conditions such as convexity and compactness, there are only a few situations in which a satisfying classification of ancient solutions is known. For example, when the ambient space is the Euclidean plane, $\mathbb{R}^2$, the only convex, compact ancient solutions are (modulo rigid motions, time translations and paraboilic dilations) the shrinking circle and the *Angenent oval* (a.k.a. the *paperclip* [4]) [9]. The Angenent oval is the family of curves $\{\Lambda_t\}_{t<0}$ defined by $\Lambda_t := \{(x,y) \in \mathbb{R}^2 : \cos x = \mathrm{e}^t \cosh y\}$.

When the ambient space is a round sphere, S^{n+1}, the situation is particularly nice: there exists no analogue of the Angenent oval and, indeed, the only geodesically convex, compact ancient solutions (in all dimensions) are the 'shrinking hemispheres' [8, 18].

Observe that the shrinking circle is *entire* — it sweeps out all of space (equivalently, its arrival time is an entire function) — whereas

[1]We refer to a solution $\{M_t^n\}_{t\in I}$ to mean curvature flow as *compact, convex, embedded, etc* if this is the case for each time slice M_t^n.

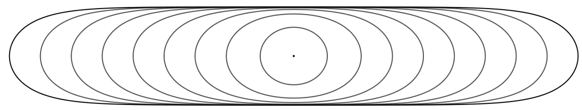


FIGURE 1. The Angenent oval solution to curve shortening flow. It shrinks to a round point as $t \to 0$ and sweeps out a strip of width π as $t \to -\infty$.

the Angenent oval lies in the strip $\{(x,y) \in \mathbb{R}^2 : |x| < \frac{\pi}{2}\}$ at all times. The following remarkable dichotomy was proved by X.-J. Wang [22].

Theorem 2.1 (X.-J. Wang's dichotomy for ancient solutions [22])**.** *Every convex, compact ancient solution to mean curvature flow is either* entire *(i.e. it sweeps out all of space) or lies at all times in a stationary slab (the region bounded by two parallel hyperplanes).*

We shall refer to a compact, convex ancient solution to mean curvature flow as an *ancient ovaloid* if it is entire or an *ancient pancake* if it lies in a slab.

In Euclidean ambient space of dimension $n+1 \geq 3$ the situation is a great deal more subtle than the planar case. Indeed, it is now known that there exist many compact, convex ancient solutions besides the shrinking spheres. In particular, for each $k \in \{0, \ldots, n-1\}$ there exists an ancient ovaloid with $O(k) \times O(n+1-k)$-symmetry which contracts to a round point as t approaches zero but becomes more eccentric as t approaches minus infinity, resembling a *shrinking cylinder* $\mathbb{R}^k \times S^{n-k}_{\sqrt{-2(n-k)t}}$ in the 'parabolic' region and a convex, translating solution in the 'tip' region. These solutions were discovered by White [24] and constructed by Wang [22] and Haslhofer–Hershkovits [11].

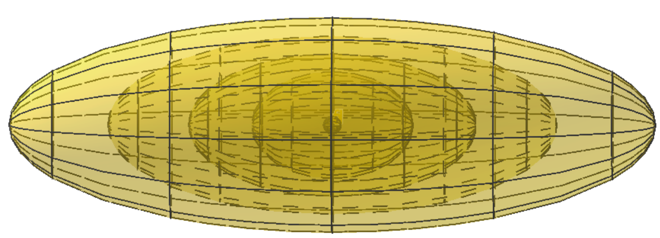


FIGURE 2. An ancient ovaloid in $\mathbb{R}^3$. It shrinks to a round point as $t \to 0$ and sweeps out all of space as $t \to -\infty$, resembling a shrinking cylinder in $B_{\sqrt{-2t}}$.

It is possible that the examples just described are the only ancient ovaloids, at least when $n = 2$. Recently, Angenent, Daskalopoulos and Šešum have proved that there is only one ancient ovaloid which is uniformly two-convex and noncollapsing [2, 3]. It remains an interesting open problem whether the ancient ovaloids are unique even amongst $O(k)\times O(n+1-k)$-symmetric solutions and it is not yet clear whether ancient ovaloids necessary carry these symmetries. It is also unclear whether or not ancient ovaloids are necessarily noncollapsing. If so, the result of Angenent, Daskalopoulos and Šešum settles the issue when $n = 2$.

The Angenent oval provides an example of an ancient pancake in $\mathbb{R}^2$. As t goes to minus infinity, the Angenent oval tends to the boundary of the strip, whereas, after translating one of its two points of maximal displacement to the origin, it resembles the translating Grim Reaper solution. In higher dimensions, Xu-Jia Wang has constructed ancient pancakes in $\mathbb{R}^{n+1}$ by taking a limit of solutions to the Dirichlet problem for the level set flow [22].

Recently, we have provided a different construction of an $O(1) \times O(n)$-invariant ancient pancake, including a precise description of its asymptotics. Our methods are rather different from Wang's.

Theorem 2.2 (Existence of ancient pancakes [5] (cf. [22]))**.** *There exists a compact, convex, $O(1)\times O(n)$-invariant ancient solution $\{M_t^n\}_{t\in(-\infty,0)}$ to mean curvature flow in $\mathbb{R}^{n+1}$ which lies in the stationary slab $\Sigma := \{x \in \mathbb{R}^{n+1} : |x_1| < \frac{\pi}{2}\}$ and has the following properties.*

- *(1 a) $\{\lambda M_{\lambda^{-2}t}\}_{t\in(-\infty,0)}$ converges uniformly in the smooth topology to the shrinking sphere $S^n_{\sqrt{-2nt}}$ as $\lambda \to 0$,*
- *(1 b) $\{M_{t+s}\}_{t\in(-\infty,-s)}$ converges locally uniformly in the smooth topology to the stationary solution $\partial\Sigma$ as $s \to -\infty$, and*
- *(1 c) for any unit vector $e \in \{e_1\}^\perp$, $\{M_{t+s} - P(e, s)\}_{t\in(-\infty,-s)}$ converges locally uniformly in the smooth topology as $s \to -\infty$ to the Grim hyperplane which translates with unit speed in the direction e, where, given any $v \in S^n$, $P(v, t)$ denotes the unique point of M_t^n with outward pointing unit normal v.*

Moreover, as $t \to -\infty$,

- *(2 a) $\min\limits_{M_t} H = H(P(e_1, t)) \leq o\left(\frac{1}{(-t)^k}\right)$ for any $k > 0$,*
- *(2 b) $\min\limits_{p\in M_t} |p| = |P(e_1, t)| \geq \frac{\pi}{2} - o\left(\frac{1}{(-t)^k}\right)$ for any $k > 0$ and*
- *(3 a) $\max\limits_{M_t} H = H(P(\varphi, t)) \geq \left(1 + \frac{n-1}{-t} + o\left(\frac{1}{(-t)^{2-\varepsilon}}\right)\right)$ for any unit vector $\varphi \in \{e_1\}^\perp$ and any $\varepsilon > 0$, and*

(3 b) $\max_{p\in M_t}|p| = |P(\varphi,t)| = -t + (n-1)\log(-t) + C + o(1)$ *for any unit vector* $\varphi \in \{e_1\}^\perp$*, where* $C \in \mathbb{R}$ *is some constant.*

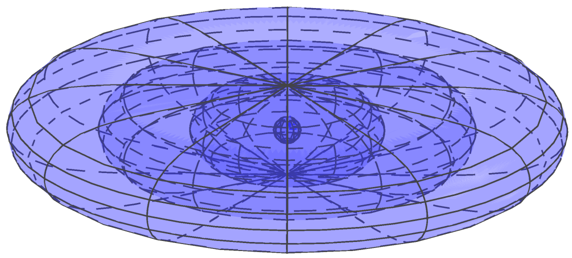


FIGURE 3. The rotationally symmetric ancient pancake. It shrinks to a round point as $t \to 0$ and sweeps out a slab of width π as $t \to -\infty$.

Existence of the solution is proved by evolving rotated timeslices Λ_{-R} of the Angenent oval and studying the evolution equation

$$\partial_t \gamma_R = -(\kappa_R + (n-1)\lambda_R)\nu_R$$

for the profile curve γ_R, where ν_R is the outward pointing unit normal to γ_R, κ_R is its curvature and $\lambda_R := -\frac{\cos\theta_R}{y}$, where θ_R is the angle the tangent vector makes with the x-axis. Let us briefly sketch the ideas involved. First, we apply the maximum principle to obtain a Sturmian type lemma: The mean curvature $H_R = \kappa_R + (n-1)\lambda_R$ always has exactly four critical points along γ_R. The maximum occurs when $\theta_R = 0$ and the minimum when $\theta_R = \frac{\pi}{2}$ (note that the double reflection symmetry of Λ_{-R} is preserved under the flow). The area A_R enclosed by γ_R satisfies

$$\frac{d}{dt}A_R = \int H_R\, ds_R = \int (\kappa_R + (n-1)\lambda_R)\, ds_R = 2\pi + (n-1)\int \frac{\lambda_R}{\kappa_R}\, d\theta_R\,,$$

where s_R is the arc-length along γ_R. By Huisken's theorem [13], γ_R contracts to a point at the final time, which we take to be zero. So integrating and applying the crude estimate $0 \le \lambda_R \le \kappa_R$ yields

$$-2\pi t \le A_R(t) \le -2n\pi t\,.$$

Since the area enclosed by the initial curve, Λ_{-R}, is $A_R(-T_R) = 2\pi R$, this gives a uniform estimate for the time interval of existence $[-T_R, 0)$. In particular, $T_R \to \infty$ as $R \to \infty$. By the compactness theory for mean curvature flow, it remains to obtain a uniform estimate for $H^R_{\max}(t) := \max H_R(\cdot,t)$ at each time. In fact, it suffices to bound

$H^R_{\min}(t) := \min H_R(\cdot, t)$: By convexity, the maximal and minimal displacements ℓ_R and h_R, respectively, satisfy

$$2h_R\ell_R \leq A_R \leq 4h_R\ell_R$$

and hence

$$-\pi t \leq 2h_R\ell_R \leq -2n\pi t\,. \tag{2}$$

These are related to the largest and smallest curvatures by

$$\frac{dh_R}{dt} = -H^R_{\min} \quad\text{and}\quad \frac{d\ell_R}{dt} = -H^R_{\max}\,.$$

So an upper bound for the speed $H^R_{\min}$ implies a lower bound for the displacement h_R, which implies an upper bound for ℓ_R via (2). An upper bound for $H_{\max}$ can then be obtained from Hamilton's Harnack inequality [1, 10].

The required upper bound for $H^R_{\min}$ follows from a simple geometric argument (see Figure 4): Since $H^R_{\min}$ occurs when $\theta = \frac{\pi}{2}$, it can be shown that the circle tangent to γ_R at $\theta = \frac{\pi}{2}$ (point p in Figure 4) which passes through $\theta = 0$ (point q in Figure 4) lies locally inside γ_R at $\theta = \frac{\pi}{2}$. We conclude that

$$H^R_{\min} \leq \frac{2nh_R}{\ell_R^2 + h_R^2}\,. \tag{3}$$

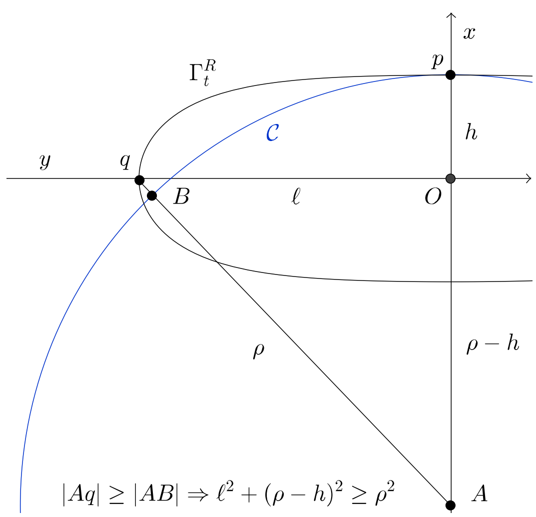


FIGURE 4. Bounding $H^R_{\min}$.

So we have uniform speed and displacement bounds for γ_R at each fixed time. Since $T_R \to \infty$ as $R \to \infty$, we can extract a compact, weakly convex ancient limit solution along a sequence $R_i \to \infty$. In fact, since the limit solution is compact, the splitting theorem for mean curvature flow ensures that the limit is strictly convex.

The convergence to a 'round point' in item (1 a) is a consequence of Huisken's theorem [13] and well-known arguments show that the 'parabolic' region converges to the boundary of the slab, as in item (1 b). With regards to item (1 c), well-known arguments also show that the 'edge' region converges to a Grim hyperplane; however, it is non-trivial to rule out limit Grim hyperplanes which are smaller than the one asymptotic to the boundary of the slab. The key step is a refined estimate for the enclosed area of the limit solution, which is obtained by deriving a better estimate for the integral of λ. We then argue that, if the limit Grim hyperplane is too thin, then the radius, and hence enclosed area must actually be quite large, contradicting the aforementioned area estimate.

The remaining asymptotics are obtained by a novel iteration argument which makes use of a refined version of the estimate (3) obtained by varying the point B in Figure 4.

These asymptotics are actually derived for *any* compact, convex, $O(n)$-invariant ancient solution contained in the slab Σ (and no smaller slab). By applying an Alexandrov reflection argument, we are then able to also obtain the following uniqueness result.

Theorem 2.3 (Uniqueness of the rotationally symmetric ancient pancake [5])**.** *Let $\{M_t\}_{t\in(-\infty,0)}$ be a compact, convex, $O(n)$-invariant ancient solution of mean curvature flow in $\mathbb{R}^{n+1}$ which lies in a slab $\Sigma_{e,\alpha} := \{x \in \mathbb{R}^{n+1} : |x \cdot e| < \alpha\}$ for some $e \in S^n$ and $\alpha > 0$ and in no smaller slab. Then, after a rigid motion and a parabolic rescaling, $\{M_t\}_{t\in(-\infty,0)}$ coincides with the solution constructed in Theorem 2.2.*

Note that reflection symmetry across the midplane of the slab is not assumed in Theorem 2.3. Moreover, by Wang's dichotomy, it suffices to assume that the solution only lies in a halfspace rather than a slab.

Note also that the arguments apply (and, indeed, are significantly simplified) in case $n = 1$. Combined with Wang's uniqueness result for the shrinking circle [22], this yields a nice new proof of the Daskalopoulos–Hamilton–Šešum classification of compact, convex ancient solutions of the curve shortening flow [9] (see [6]).

Unlike the ancient ovaloids, ancient pancakes cannot possess $O(k) \times O(n+1-k)$-symmetry for any $k > 1$ since such a solution could be enclosed by a cylinder $S^1_R \times \mathbb{R}^{n-1}$ for some sufficiently large R (uniformly in time), contradicting the avoidance principle. So our result gives a complete classification of doubly symmetric ancient pancakes.

3. Pancakes with discrete symmetry groups

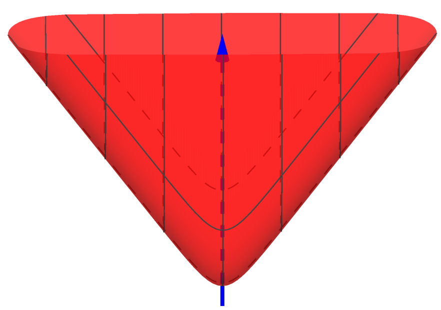
There may exist non-rotationally symmetric ancient pancake solutions and in order to construct and classify further examples, we have also studied the existence and geometric properties of translators [7].

Inspired by the work of Shariyari [20] and Spruck–Xiao [21], we were able to obtain a complete resolution to the existence question in all dimensions.

Theorem 3.1 (Existence of convex translators in all admissible slabs [7])**.** *For every $n \geq 2$ and every $\theta \in (0, \frac{\pi}{2})$ there exists a strictly convex translator W^n_θ which lies in*

$$\Sigma^{n+1}_\theta := \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : |x| < \tfrac{\pi}{2} \sec\theta\} \subset \mathbb{R}^{n+1}$$

and in no smaller slab.



The 'flying wing' W^2_θ of [7] with $\theta = \frac{\pi}{4}$ (right). The translation direction is vertical.

Around the same time our work was completed, Hoffman, Ilmanen, Martìn and White provided an existence theorem for all slabs of width greater than π in the case $n = 2$ [12, Theorem 1.1]. They were also able to prove uniqueness in this case, thereby completing the classification of translating graphs in $\mathbb{R}^3$. Finally, they gave a different construction of examples of translating graphs in slabs in $\mathbb{R}^{n+1}$, extending an earlier construction of Ilmanen for the case $n = 2$ [12]. These solutions are parametrized by the vector of principal curvatures at the 'tip' (the unique point at which the downward unit normal is $-e_n$).

From our aforementioned study of the translators with $O(1) \times O(n-1)$-symmetry, the following conjecture appears natural.

Conjecture 3.2 (Dihedral pancakes)**.** *Given any $k \geq 3$ there exists an ancient pancake lying in the slab $(-\frac{\pi}{2}\sec\frac{\pi}{k}, \frac{\pi}{2}\sec\frac{\pi}{k}) \times \mathbb{R}^2 \subset \mathbb{R}^3$ (and* *in no smaller slab) with symmetry group $O(1) \times D_k$, where D_k is the symmetry group of the regular k-sided polygon. Modulo translations and rotations, this is the unique such solution. Let $\{\phi_i\}_{i=1}^k \subset \mathbb{C} \cong \{0\} \times \mathbb{R}^2$ be the k-th roots of unity in the $\{x_1 = 0\}$ plane. Up to a rotation, the solution has the following asymptotics: Given a unit vector $\phi \in \{0\} \times \mathbb{R}^2$, the asymptotic translator in the ϕ-direction is the oblique Grim plane $\Gamma^2_{\frac{\pi}{k}}$, except when $\phi \in \{\phi_i\}_{i=1}^k$, in which case the asymptotic translator is the flying wing translator $W^2_{\frac{\pi}{k}}$.*

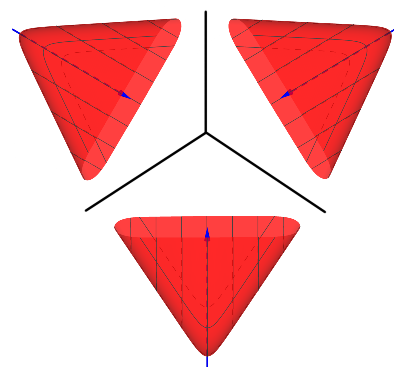


FIGURE 5. Gluing three flying wing translators at infinity to form a compact ancient solution.

When $n = 2$, it is concievable that these, along with the rotationally symmetric one, are the only ancient pancakes.

Generalizing these principles leads to the following natural question.

Question 3.3. *Do there exist ancient pancakes in $\mathbb{R}^{n+1}$ with symmetry groups $O(1) \times G^n$, where G^n is the symmetry group of a regular n-polytope?*

REFERENCES

[1] ANDREWS, B. Harnack inequalities for evolving hypersurfaces. *Math. Z. 217*, 2 (1994), 179–197.

[2] ANGENENT, S., DASKALOPOULOS, P., AND SESUM, N. Unique asymptotics of ancient convex mean curvature flow solutions. Preprint, arXiv:1503.01178v3.

[3] ANGENENT, S., DASKALOPOULOS, P., AND SESUM, N. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. Preprint, arXiv:1804.07230.

[4] BAKAS, I., AND SOURDIS, C. Dirichlet sigma models and mean curvature flow. *Journal of High Energy Physics 2007*, 06 (2007), 057.

[5] Bourni, T., Langford, M., and Tinaglia, G. Collapsing ancient solutions of mean curvature flow. Preprint available at arXiv:1705.06981.

[6] Bourni, T., Langford, M., and Tinaglia, G. A new geometric proof of the characterization of the angenent oval. Work in progress.

[7] Bourni, T., Langford, M., and Tinaglia, G. On the existence of translating solutions of mean curvature flow in slab regions. Preprint available at arxiv.org/abs/1805.05173.

[8] Bryan, P., and Louie, J. Classification of convex ancient solutions to curve shortening flow on the sphere. *J. Geom. Anal. 26*, 2 (2016), 858–872.

[9] Daskalopoulos, P., Hamilton, R., and Sesum, N. Classification of compact ancient solutions to the curve shortening flow. *J. Differential Geom. 84*, 3 (2010), 455–464.

[10] Hamilton, R. S. Harnack estimate for the mean curvature flow. *J. Differential Geom. 41*, 1 (1995), 215–226.

[11] Haslhofer, R., and Hershkovits, O. Ancient solutions of the mean curvature flow. *Commun. Anal. Geom. 24*, 3 (2016), 593–604.

[12] Hoffman, D., Ilmanen, T., Martin, F., and White, B. Graphical translators for mean curvature flow. Preprint, arXiv:1805.10860.

[13] Huisken, G. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom. 20*, 1 (1984), 237–266.

[14] Huisken, G. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom. 31*, 1 (1990), 285–299.

[15] Huisken, G. Local and global behaviour of hypersurfaces moving by mean curvature. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, vol. 54 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1993, pp. 175–191.

[16] Huisken, G., and Sinestrari, C. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math. 183*, 1 (1999), 45–70.

[17] Huisken, G., and Sinestrari, C. Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differential Equations 8*, 1 (1999), 1–14.

[18] Huisken, G., and Sinestrari, C. Convex ancient solutions of the mean curvature flow. *J. Differential Geom. 101*, 2 (2015), 267–287.

[19] Langford, M. A general pinching principle for mean curvature flow and applications. *Calculus of Variations and Partial Differential Equations 56*, 4 (Jul 2017), 107.

[20] Shahriyari, L. Translating graphs by mean curvature flow. *Geom. Dedicata 175* (2015), 57–64.

[21] Spruck, J., and Xiao, L. Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature. Preprint, arXiv:1703.01003.

[22] Wang, X.-J. Convex solutions to the mean curvature flow. *Ann. of Math. (2) 173*, 3 (2011), 1185–1239.

[23] White, B. The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc. 13*, 3 (2000), 665–695 (electronic).

[24] White, B. The nature of singularities in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc. 16*, 1 (2003), 123–138 (electronic).