

Assignment 2

1. Using the method of separation of variables, find the solution to the problem

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } B \times (0, \infty) \\ u = 0 & \text{along } \partial B \times (0, \infty) \\ u = \varphi & \text{at } B \times \{0\} \end{cases}$$

in terms of spherical harmonics, where B is the unit ball in $\mathbb{R}^n$ and $\varphi \in C^\infty(\overline{B})$.
Hint: consider solutions of the form $u(x, t) = T(t)R(r)\Theta(z)$.

2. (Galilean invariance)

- (a) Show that the fundamental solution to the heat equation on $\mathbb{R}^n$ is invariant under all Galilean boosts.
- (b) Find all solutions $u \in \mathcal{C}^{2;1}(\mathbb{R} \times (0, \infty))$ to the heat equation on $\mathbb{R}$ which are invariant under the Galilean boosts.

3. Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function.

- (a) Using the mean value property and the divergence theorem, prove that

$$\sup_{B_r(x)} |Du| \leq \frac{C_n}{r} \sup_{\partial B_r(x)} |u|.$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where C is a constant which depends only on n (in particular, not on u). *Hint: $D_j u = \operatorname{div}(u e_j)$.*

- (b) Deduce that every harmonic function $u \in C^2(\mathbb{R}^n)$ satisfying

$$|u(x)| \leq o(|x|) \text{ as } x \rightarrow \infty$$

is constant. (The “little-o” notation means that $\frac{|u(x)|}{|x|} \rightarrow 0$ as $x \rightarrow \infty$. Note that this is weaker than $|u(x)| \leq O(1)$ as $x \rightarrow \infty$, meaning that $\limsup_{x \rightarrow \infty} \frac{|u(x)|}{1} < \infty$. I.e. u is bounded.)

- (c) Deduce that any harmonic function $u \in C^2(\mathbb{R}^n)$ satisfying

$$|D^k u(x)| \leq o(|x|)$$

is a harmonic polynomial of degree at most k .

- (d) Show that every positive harmonic function $u \in C^2(\mathbb{R}^n)$ is bounded. *Hint: apply the Harnack inequality in large balls B_r .*
- (e) Deduce that any positive harmonic function $u \in C^2(\mathbb{R}^n)$ is constant.

Assignment Due: Friday, 14th August by 17:00