

Assignment 3

1. Provide an example of

(a) a function $u : [0, 1] \rightarrow \mathbb{R}$ which is in $C^\alpha([0, 1])$ but not in $C^\beta([0, 1])$ for any $\beta > \alpha$.

(b) a function $u : [0, 1] \rightarrow \mathbb{R}$ which is in $C^{1,1}([0, 1])$ but not in $C^2([0, 1])$.

2. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is continuously differentiable. Show that f is α -Hölder continuous for all $\alpha \in (0, 1]$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Show, for any choice of $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $u, v \in C^\alpha(\Omega)$, $\alpha \in (0, 1]$, show that

$$[uv]_{C^\alpha(\Omega)} \leq \sup_{\Omega} |u| [v]_{C^\alpha(\Omega)} + \sup_{\Omega} |v| [u]_{C^\alpha(\Omega)}.$$

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $\alpha, \beta \in (0, 1]$ with $\beta \leq \alpha$, show that

(a) $C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})$.

(b) the inclusion map $\iota : C^\alpha(\overline{\Omega}) \rightarrow C^\beta(\overline{\Omega})$ is continuous.

(c) the embedding $\iota : C^\alpha(\overline{\Omega}) \rightarrow C^\beta(\overline{\Omega})$ is *compact* (i.e., bounded subsets of $C^\alpha(\overline{\Omega})$ are relatively compact in $C^\beta(\overline{\Omega})$) if $\beta < \alpha$. *Hint: by the Arzelà–Ascoli theorem, a bounded sequence $\{u_j\}_{j \in \mathbb{N}} \subset C^\alpha(\overline{\Omega})$ admits a subsequence which converges in the uniform topology to some limit u . Show that $v_j \doteq u_j - u$ converges to zero in the β -Hölder topology.*

6. Suppose that the coefficients $(a, b, c) : \Omega \rightarrow S^{n \times n} \times \mathbb{R}^n \times \mathbb{R}$ and the function $f : \Omega \rightarrow \mathbb{R}$ are of class $C^{k,\alpha}$ (resp. C^∞) and the boundary datum ϕ is of class $C^{k+2,\alpha}$ (resp. C^∞). Suppose that $u \in C^2(\Omega)$ satisfies $-Lu = f$ in Ω . Show that $u \in C^{k+2,\alpha}(B)$ (resp. $C^\infty(B)$) for every ball $B \Subset \Omega$. *Hint 1: If u is of class C^{j+2} for some $j \leq k$, then the equation $-Lu = f$ may be differentiated j times. Hint 2: Given a function $v : \Omega \rightarrow \mathbb{R}$, a unit coordinate direction e_ℓ , $x \in B$, and sufficiently small h , consider the difference quotient*

$$\delta_\ell^h v(x) \doteq \frac{v(x + he_\ell) - v(x)}{h}.$$

Assignment Due: Friday, 4th September by 17:00