

Math4202: Theory of partial differential equations

Week 6: Perron's method

September 22, 2026

Perron's method

We are now ready to complete the proof of our existence theorem, which we now recall.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,\alpha}$ and suppose that the coefficients $(a, b, c) : \overline{\Omega} \rightarrow S^{n \times n} \times \mathbb{R}^n \times \mathbb{R}$ are α -Hölder continuous, a is positive definite, and $c \leq 0$. If $f \in C^\alpha(\overline{\Omega})$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$, then the Dirichlet problem*

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \quad (1)$$

admits a unique solution in $C^{2,\alpha}(\overline{\Omega})$.

We achieve this using PERRON'S METHOD, which reduces solubility of an equation in a given domain to its solubility in small balls (which we have established using the method of continuity and the solubility, in $C^{2,\alpha}$, of Poisson's equation over a ball).

The key tool is the maximum principle, which implies that any subsolution taking the same boundary values as a solution u necessarily lies below u . If there exists at least one subsolution, then the "largest" subsolution exists. If this happens to be smooth, then it must actually be a solution.

Let us recall the following estimate.

Proposition 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\text{diam}(\Omega) \leq R$. Suppose that the coefficients $(a, b, c) : \Omega \rightarrow S^{n \times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy*

$$a^{ij}(x) \geq \lambda \delta^{ij}, \quad c \leq 0, \quad \text{and} \quad \lambda^{-2}|b(x)|^2 - \lambda^{-1}c(x) \leq \Lambda.$$

If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies

$$-(a^{ij}u_{ij} + b^i u_i + cu) \leq f \quad \text{in } \Omega,$$

then

$$\sup_{\Omega} u \leq \max_{\partial\Omega} u_+ + C \sup_{\Omega} \frac{f_+}{\lambda},$$

where $C \doteq e^{|\xi|R}$.

Barrier subsolutions

Consider, then, the linear elliptic operator

$$L \doteq a^{ij}D_i D_j + b^i D_i + c.$$

We assume throughout this section that the operator L satisfies the hypotheses of Theorem 1. In particular, L adheres to the maximum

¹ We will also see that existence (and uniqueness) can be established in the larger space $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ if the boundary data are merely continuous and $\partial\Omega$ satisfies the exterior ball condition.

principle and the strong maximum principle. We shall, in addition, assume a *local* solubility condition for Dirichlet problems corresponding to L (to be made precise below). This condition may be verified independently for Poisson's equation (and hence, by the method of continuity, for all L satisfying our hypotheses).

We say that a function $v : \Omega \rightarrow \mathbb{R}$ satisfies $-Lv \leq f$ (or that v is a subsolution to the equation $-Lu = f$) **IN THE BARRIER SENSE** if, for every $\Omega' \Subset \Omega$ and $\varphi \in C^2(\Omega')$ with $-L\varphi = f$ in Ω' ,

$$v \leq \varphi \text{ on } \partial\Omega' \implies v \leq \varphi \text{ in } \Omega'.$$

Since L adheres to the maximum principle, any classical subsolution $v \in C^2(\Omega)$ is a subsolution in the barrier sense.

If f is continuous and we are able to solve the Dirichlet problem for the equation $-Lu = f$ in C^2 for sufficiently small balls about any point of Ω , then the converse is also true; that is, every barrier subsolution $v \in C^2(\Omega)$ is a classical subsolution. To see this, suppose, to the contrary, that $-Lv(x_0) > f(x_0)$ at some point $x_0 \in \Omega$ for some barrier subsolution v . Since f and Lv are continuous, we can find $r > 0$ such that $-Lv > f$ on $B_r(x_0)$ (in the classical sense). If r is sufficiently small, then, by hypothesis, we may solve the boundary value problem

$$\begin{cases} -L\varphi = f & \text{in } B_r(x_0) \\ \varphi = v & \text{on } \partial B_r(x_0) \end{cases}$$

in $C^2(B_r(x_0)) \cap C^0(\overline{B_r(x_0)})$. So the maximum principle implies that $v \geq \varphi$ in $B_r(x_0)$. On the other hand, $v \leq \varphi$ since v is a subsolution, so we conclude that $v \equiv \varphi$. But then $-Lv \equiv f$, in contradiction with our assumption.

In fact, we will assume a slightly stronger local solubility condition². Namely, for fixed $\alpha \in (0, 1)$, bounded domain $\Omega \subset \mathbb{R}^n$, and right hand side $f \in C^\alpha(\Omega)$, we assume that we can find, for every $x \in \Omega$, some $r > 0$ and some $\delta \in (0, 1)$ such that $B_r(x) \Subset \Omega$ and the Dirichlet problem

$$\begin{cases} -Lu = f & \text{in } B_{\delta r}(y) \\ u = \phi & \text{on } \partial B_{\delta r}(y) \end{cases} \quad (2)$$

is soluble in $C^{2,\alpha}(B_{\delta r}(y)) \cap C^0(\overline{B_{\delta r}(y)})$ over every $B_{\delta r}(y) \subset B_r(x)$ for every $\phi \in C^0(\partial B_{\delta r})$.

We may define barrier supersolutions analogously, and these will satisfy corresponding supersolution properties.

Next observe that barrier sub- and supersolutions admit a comparison principle.

Proposition 3. Suppose that $u, v \in C^0(\overline{\Omega})$ satisfy, respectively, $-Lu \leq f$ and $-Lv \geq f$ in the barrier sense. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω .

² The following local solubility condition will suffice to obtain global solutions with $C^{2,\alpha}$ interior regularity. To obtain $C^{2,\alpha}$ regularity up to the boundary for the global solutions, we will need to make a corresponding local solubility assumption.

Proof. Set $M \doteq \sup_{\Omega}(u - v)$ and suppose, contrary to the claim, that $M > 0$. Since $u - v \leq 0$ on $\partial\Omega$, we can find $x_0 \in \Omega$ and $r > 0$ such that $u(x_0) - v(x_0) = M$ and $u - v \not\equiv M$ on $\partial B_r(x_0)$. In fact, we can also arrange that r is as small as we like, so that we may find solutions $\bar{u}, \bar{v} \in C^2(B_r(x_0)) \cap C^0(\bar{B}_r(x_0))$ to the Dirichlet problem for L in $B_r(x_0)$ with respective boundary data u and v on $\partial B_r(x_0)$. Note that $u \leq \bar{u}$ and $v \geq \bar{v}$ in $B_r(x_0)$. So the maximum principle implies that

$$M = (u - v)(x_0) \leq (\bar{u} - \bar{v})(x_0) \leq \max_{\partial B_r(x_0)} (\bar{u} - \bar{v}) = \max_{\partial B_r(x_0)} (u - v) \leq M,$$

and hence $(\bar{u} - \bar{v})(x_0) = \max_{\bar{B}_r(x_0)} (\bar{u} - \bar{v})$. That is, $\bar{u} - \bar{v}$ attains its maximum at an interior point. But this violates the strong maximum principle since $\bar{u} - \bar{v}$ is not constant. $\square$

We need one last lemma.

Lemma 4. *If $u, v \in C^0(\bar{\Omega})$ are both barrier subsolutions to the equation $-L\phi = f$, then the function $\max\{u, v\}$ is also a subsolution.*

Proof. Exercise. $\square$

Consider now the Dirichlet problem

$$\begin{cases} -Lu = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases} \quad (3)$$

Define the PERRON (SUB)SOLUTION $u : \Omega \rightarrow \mathbb{R}$ by

$$u(x) \doteq \sup\{v(x) : v \in C^0(\bar{\Omega}), -Lv \leq f \text{ in } \Omega, \text{ and } v \leq \phi \text{ on } \partial\Omega\}.$$

We shall call a function $v \in C^0(\bar{\Omega})$ satisfying $-Lv \leq f$ in Ω (in the barrier sense) and $u \leq \phi$ on $\partial\Omega$ a (BARRIER) SUBSOLUTION TO THE DIRICHLET PROBLEM (3).

Observe that u takes finite values. Indeed, choosing ξ so that

$$\lambda \doteq a^{ij}\xi_i\xi_j + b^i\xi_i + c > 0$$

and setting $\bar{d} \doteq \sup_{x \in \Omega} (\xi \cdot x)$ and $\underline{d} \doteq \inf_{x \in \Omega} \xi \cdot x$, we find that the function $\underline{u} \in C^0(\bar{\Omega})$ defined by

$$\underline{u}(x) \doteq -\sup_{\partial\Omega} (-\phi)_+ - e^{-\underline{d}} \left(e^{\bar{d}} - e^{x \cdot \xi} \right) \sup_{\Omega} \frac{(-f)_+}{\lambda}$$

satisfies $-L\underline{u} \leq f$ in Ω and $\underline{u} \leq \phi$ on $\partial\Omega$. Thus (by definition of u) $u \geq \underline{u}$ in Ω and hence

$$\inf_{\Omega} u \geq -\sup_{\partial\Omega} (-\phi)_+ - e^{|\xi|R} \sup_{\Omega} \frac{(-f)_+}{\lambda}, \quad (4)$$

where $R \geq \text{diam } \Omega$. On the other hand, since the function

$$\bar{u}(x) \doteq \sup_{\partial\Omega} \phi_+ + e^{-d} \left(e^{\bar{d}} - e^{x \cdot \xi} \right) \sup_{\Omega} \frac{f_+}{\lambda}$$

is a supersolution to the Dirichlet problem, Proposition 3 yields

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} \phi_+ + e^{|\xi|R} \sup_{\Omega} \frac{f_+}{\lambda}.$$

We conclude that

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f|, \quad (5)$$

where C depends only on n , R , and the coefficients of L .

Smoothness of the Perron (sub)solution in the interior

We claim that $u \in C^{2,\alpha}(\Omega)$ and $-Lu = f$. To see this, fix some $x \in \Omega$ and let $u_j \in C^0(\bar{\Omega})$ form a sequence of subsolutions to (3) such that $u_j(x) \rightarrow u(x)$. Given $r \leq 1$ sufficiently small, we can, by hypothesis, find a function $U_j \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(B_r(x))$ satisfying $U_j \equiv u_j$ on $\Omega \setminus B_r(x)$ and

$$\begin{cases} -LU_j = f & \text{in } B_r(x) \\ U_j = u_j & \text{on } \partial B_r(x). \end{cases}$$

The interior Schauder estimate, Proposition 2 and the estimate (5) yield

$$|U_j|_{C^{2,\alpha}(B_{r/2}(x))} \leq C \left(\sup_{\partial\Omega} |\phi| + |f|_{C^\alpha(\Omega)} \right),$$

where C depends only on n , α , and the coefficients of L . So the Arzelà–Ascoli theorem implies that a subsequence of the functions U_j converges in $C^2(B_{r/2}(x))$ to some limit $U \in C^{2,\alpha}(B_{r/2}(x))$ satisfying $-LU = f$ in $B_{r/2}(x)$. Moreover, since $u_j(x) \rightarrow u(x)$ and (by Proposition 3) $u_j \leq U_j$, we find that $U(x) \geq u(x)$. On the other hand, by the definition of u , $U \leq u$ in $B_{r/2}$. So $U(x) = u(x)$.

We claim that $U \equiv u$ in $B_{\delta r}(x)$ for some $\delta > 0$. Suppose, to the contrary, that for any $\delta > 0$ we can find $y \in B_{\delta r}(x)$ such that $U(y) < u(y)$. By the definition of u , we can find a subsolution $v \in C^0(\bar{\Omega})$ to (3) such that $U(y) < v(y) \leq u(y)$. Set $v_j \doteq \max\{U_j, v\}$. For δ sufficiently small, we can find a function $V_j \in C^0(\bar{\Omega})$ satisfying $V_j \equiv v_j$ on $\Omega \setminus B_{4\delta r}(y)$ and

$$\begin{cases} -LV_j = f & \text{in } B_{4\delta r}(y) \\ V_j = v_j & \text{on } \partial B_{4\delta r}(y). \end{cases}$$

Since $U_j \leq V_j$ on $\partial B_{4\delta r}$, we have $U_j \leq V_j$ in $B_{4\delta r}(y)$. Since V_j is a subsolution to the Dirichlet problem (3), we also have $V_j \leq u$ by

the definition of u . Since v is a subsolution and $v \leq V_j$ on $\partial B_{4\delta r}$, we also have $v \leq V_j$ in $B_{4\delta r}$. In particular, $\lim_{j \rightarrow \infty} V_j(x) = u(x)$, and $U(y) < \lim_{j \rightarrow \infty} V_j(y) \leq u(y)$.

Since $V_j \equiv v_j \geq U_j$ on $\Omega \setminus B_{4\delta r}(y)$, the interior Schauder estimate, Propositions 2 and 3 and the estimate (5) yield, for j sufficiently large,

$$|V_j|_{C^{2,\alpha}(B_{2\delta r}(y))} \leq C \left(\sup_{\partial\Omega} |\phi| + |f|_{C^\alpha(\Omega)} \right),$$

where C depends on n , α , and the coefficients of L . So the Arzelà–Ascoli theorem implies that a subsequence of the functions V_j converges in $C^2(B_{2\delta r}(y))$ to some limit $V \in C^{2,\alpha}(B_{2\delta r}(y))$ satisfying $-LV = f$ and $U \leq V \leq u$ in $B_{2\delta r}(y)$, and $U(y) < V(y)$. Since $-LU = -LV = f$ in $B_{2\delta r}(y)$, $U \leq V$ and $U(x) = V(x)$ at $x \in B_{\delta r}(y)$, the strong maximum principle implies that $U \equiv V$ in $B_{2\delta r}(y)$, in contradiction with the inequality $U(y) < V(y)$.

We conclude that $U \equiv u$ in $B_{\delta r}(x)$. But then $u \in C^{2,\alpha}(B_{\delta r})$ and $-Lu = f$ in $B_{\delta r}$. The claim follows since x was arbitrary.

Continuity of the Perron solution up to the boundary

So we have found a solution $u \in C^2(\Omega)$ to the equation $-Lu = f$. However, we have not yet shown that u attains the boundary values ϕ . This will be possible so long as Ω is of class C^1 . In fact, it suffices for Ω to satisfy the EXTERIOR BALL CONDITION, which means that for each point $x_0 \in \partial\Omega$ we can find a ball $B \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $\overline{B} \cap \overline{\Omega} = \{x_0\}$.

So fix $x_0 \in \partial\Omega$. We need to show that $u(x) \rightarrow \phi(x)$ as $x \rightarrow x_0$. To do this, we shall construct upper and lower barriers for u near x_0 which take values arbitrarily close to $\phi(x_0)$ at x_0 . First observe that we can find a function $w : \Omega \rightarrow \mathbb{R}$ such that $-Lw \geq 1$ in Ω , $w > 0$ on $\partial\Omega \setminus \{x_0\}$, and $w(x_0) = 0$. Indeed, given a ball $B_r(p)$ in $\mathbb{R}^n \setminus \Omega$ such that $\overline{B_r(p)} \cap \overline{\Omega} = \{x_0\}$, we can set

$$w(x) = w_{\mu,\sigma}(x) \doteq \mu (r^{-\sigma} - |x - p|^{-\sigma})$$

for suitable $\mu > 0$ and $\sigma > 0$. To see this, observe that

$$-Lw_{\mu,\sigma} \geq \frac{\mu}{|x - p|^{\sigma+2}} \left(\sigma(\sigma + 2) a^{ij} \frac{(x - p)_i}{|x - p|} \frac{(x - p)_j}{|x - p|} - \sigma \left(\text{tr } a + b^i (x - p)_i \right) \right).$$

The claim follows from the uniform positivity of a and the bounds for a and b .

Now fix any sequence of numbers $\varepsilon_i \searrow 0$ and, for each i , choose $\delta_i > 0$ such that $|\phi(x) - \phi(x_0)| < \varepsilon_i$ for $x \in B_{\delta_i}(x_0) \cap \partial\Omega$. We may further choose $k_i > 0$ so that

$$k_i w \geq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f| + |\phi(x_0)| \quad \text{on } \partial B_{\delta_i}(x_0) \cap \Omega,$$

where C is the constant in the estimate (5). If we also ensure that $k_i \geq \sup_{\Omega} |f + c\phi(x_0)|$, then we find that the functions

$$\underline{w}_i \doteq \phi(x_0) - k_i w - \varepsilon_i \quad \text{and} \quad \overline{w}_i \doteq \phi(x_0) + k_i w + \varepsilon_i,$$

satisfy

$$\begin{cases} -L\underline{w}_i \leq 0 & \text{in } B_{\delta_i}(x_0) \cap \Omega \\ \underline{w}_i \leq \phi & \text{on } B_{\delta_i}(x_0) \cap \partial\Omega \\ \underline{w}_i \leq \underline{M} \doteq -\sup_{\partial\Omega} |\phi| - C \sup_{\Omega} |f| & \text{on } \partial B_{\delta_i}(x_0) \cap \Omega \end{cases}$$

and

$$\begin{cases} -L\overline{w}_i \geq 0 & \text{in } B_{\delta_i}(x_0) \cap \Omega \\ \overline{w}_i \geq \phi & \text{on } B_{\delta_i}(x_0) \cap \partial\Omega \\ \overline{w}_i \geq \overline{M} \doteq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f| & \text{on } \partial B_{\delta_i}(x_0) \cap \Omega. \end{cases}$$

Finally, we set

$$\underline{u}_i \doteq \begin{cases} \max\{\underline{w}_i, \underline{m}\} & \text{in } B_{\delta_i}(x_0) \cap \Omega \\ \underline{m} & \text{in } \Omega \setminus \overline{B}_{\delta_i}(x_0) \end{cases}$$

and

$$\overline{u}_i \doteq \begin{cases} \min\{\overline{w}_i, \overline{m}\} & \text{in } B_{\delta_i}(x_0) \cap \Omega \\ \overline{m} & \text{in } \Omega \setminus \overline{B}_{\delta_i}(x_0), \end{cases}$$

where

$$\overline{m} \doteq \sup_{\partial\Omega} \phi_+ + e^{-d} \left(e^{\bar{d}} - e^{x \cdot \xi} \right) \sup_{\Omega} \frac{f_+}{\lambda}$$

and

$$\underline{m} \doteq -\inf_{\partial\Omega} (-\phi)_+ - e^{-d} \left(e^{\bar{d}} - e^{x \cdot \xi} \right) \sup_{\Omega} \frac{(-f)_+}{\lambda}.$$

By Lemma 4, $\underline{u}_i$ and $\overline{u}_i$ are, respectively, sub- and supersolutions to the Dirichlet problem (3), so we find that $\underline{u}_i \leq u \leq \overline{u}_i$. Taking $i \rightarrow \infty$ then yields $u(x_0) = \phi(x_0)$. Since x_0 was arbitrary, we conclude that $u|_{\partial\Omega} = \phi$.

It remains to prove that u is of class $C^{2,\alpha}$ up to the boundary of Ω (assuming Ω is of class $C^{2,\alpha}$, $\phi \in C^{2,\alpha}(\overline{\Omega})$ and $f \in C^\alpha(\overline{\Omega})$). But let us first note that we have proved the following result.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the exterior ball condition (e.g. $\partial\Omega$ is of class C^2) and suppose that the coefficients $(a, b, c) : \Omega \rightarrow S^{n \times n} \times \mathbb{R}^n \times \mathbb{R}$ are bounded and α -Hölder continuous, a is positive definite, and $c \leq 0$. If f is bounded and α -Hölder continuous and $\phi \in C^0(\partial\Omega)$, then the Dirichlet problem*

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \quad (6)$$

admits a unique solution in $C^{2,\alpha}(\Omega) \cap C^0(\Omega)$.

Smoothness of the Perron solution up to the boundary

Finally, we prove (under the full hypotheses of Theorem 1) that $u \in C^{2,\alpha}(\bar{\Omega})$, so long as the local solutions (to the problems (2)) are of class $C^{2,\alpha}$ up to the boundary.

Fix some $x_0 \in \partial\Omega$. Since Ω is of class $C^{2,\alpha}$, we can find a neighbourhood (in $\mathbb{R}^n$) U of x_0 , an open set $V \subset \mathbb{R}^n$ containing $\bar{B}_1$, a $C^{2,\alpha}$ diffeomorphism $\Phi : U \rightarrow V$, and some $\rho > 0$ such that $B_1 \subset \Phi(\Omega)$ and $B_\rho(\tilde{x}_0) \cap \partial B_1 \subset \Phi(\partial\Omega \cap U) \subset \partial B_1$, where $\tilde{x}_0 \doteq \Phi(x_0)$.

Set $(\tilde{a}, \tilde{b}, \tilde{c}) \doteq (a, b, c) \circ \Phi^{-1}|_{\bar{B}_1}$, $\tilde{f} \doteq f \circ \Phi^{-1}|_{\bar{B}_1}$, and $\tilde{u} \doteq u \circ \Phi^{-1}|_{\bar{B}_1}$, and let $\tilde{\phi} : B_{3/2} \rightarrow \mathbb{R}$ be the radial extension of $\tilde{u}$; that is, $\tilde{\phi}(x) \doteq |x|\tilde{u}(x/|x|)$ for $x \neq 0$ and $\tilde{\phi}(0) \doteq 0$. For each $k \in \mathbb{N}$, consider $\tilde{\phi}_k \doteq \tilde{\phi} * \eta_{1/k}$, where η_ε is the standard mollifier. Note that $\tilde{\phi}_k \rightarrow \tilde{\phi}$ in the uniform topology and $|\tilde{\phi}_k|_{C^{2,\alpha}(B_\rho(\tilde{x}_0))} \leq 2|\tilde{\phi}|_{C^{2,\alpha}(B_\rho(\tilde{x}_0))}$ for k sufficiently large.

By hypothesis, we can find, for each $k \in \mathbb{N}$, a solution $\tilde{u}_k \in C^{2,\alpha}(\bar{B}_1)$ to the Dirichlet problem

$$\begin{cases} -\tilde{L}\tilde{u}_k = \tilde{f} & \text{in } B_1 \\ \tilde{u}_k = \tilde{\phi}_k & \text{on } \partial B_1, \end{cases}$$

where $\tilde{L} \doteq \tilde{a}^{ij}D_iD_j + \tilde{b}^kD_k + \tilde{c}$. It follows from the maximum principle, the interior Schauder estimate, and the boundary Schauder estimate, in conjunction with the Arzelà–Ascoli theorem, that, after passing to a subsequence, the solutions $\tilde{u}_k$ converge to some limit $\tilde{u}_\infty \in C^0(\bar{B}_1) \cap C^2(B_1) \cap C^{2,\alpha}(\bar{B}_1 \cap B_\rho(\tilde{x}_0))$ uniformly in $\bar{B}_1$, in C^2 on compact subsets of B_1 and in $C^2(\bar{B}_1 \cap B_\rho(\tilde{x}_0))$. But since $\tilde{u}_\infty$ takes the same boundary values as $\tilde{u}$, we must have $\tilde{u}_\infty \equiv \tilde{u}$. The claim follows.

Parabolic equations

Perron's method is readily adapted to parabolic equations³. We hope that you may find this at least plausible, if not convincing, as we do not have time to cover it!⁴

Exercises

Exercise 1. Prove Lemma 4.

³ In this context, it is known as the Perron–Wiener–Brelot method. See, e.g., Neil A. Watson *Introduction to Heat Potential Theory* (Mathematical Surveys and Monographs, vol. 182, American Mathematical Society, 2012).

⁴ I.e. I do not have time to write it.