

Math4202: Theory of partial differential equations

Week 5: Schauder's estimate

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Schauder's theorem

Schauder's Theorem, a fundamental result in the study of elliptic partial differential equations, states, roughly speaking, that solutions to uniformly elliptic affine linear equations are two *Hölder derivatives* more regular than their coefficients. It is considered a *perturbative result*, in that it relies on the fact that (under the conditions of the theorem) the equation closely resembles the Laplace equation at sufficiently small scales. The theorem consists of two results, the *interior estimate*, and the *boundary estimate* (which together yield a *global estimate*).

Schauder's theorem is phrased in terms of *Hölder spaces*. These spaces play a crucial role in the development of the theory of non-linear partial differential equations, so it is worth reviewing them here.

Hölder spaces

Given $\alpha \in (0, 1]$ and a bounded open set $\Omega \subset \mathbb{R}^n$, a function $u : \Omega \rightarrow \mathbb{R}$ is α -HÖLDER CONTINUOUS AT $x \in \Omega$ if

$$\sup_{y \in \Omega \setminus \{x\}} \frac{|u(y) - u(x)|}{|y - x|^\alpha} < \infty.$$

A function $u : \Omega \rightarrow \mathbb{R}$ is then called α -HÖLDER CONTINUOUS if it is α -Hölder continuous at all points, and UNIFORMLY α -HÖLDER CONTINUOUS if

$$[u]_{C^\alpha(\Omega)} \doteq \sup_{(x,y) \in \Omega \times \Omega \setminus \{(z,z): z \in \Omega\}} \frac{|u(y) - u(x)|}{|y - x|^\alpha} < \infty.$$

We simply call $u : \Omega \rightarrow \mathbb{R}$ (UNIFORMLY) HÖLDER CONTINUOUS (AT x) if it is (uniformly) α -Hölder continuous (at x) for some $\alpha \in (0, 1]$.

When we consider parabolic equations, we instead make use of the parabolic distance function,

$$\mathcal{D}((x, t), (y, s)) \doteq |x - y| + \sqrt{|t - s|}.$$

Given $\alpha \in (0, 1]$ and a bounded, (parabolically) open¹ set $Q \subset \mathbb{R}^n \times \mathbb{R}$, a function $u : Q \rightarrow \mathbb{R}$ is α -HÖLDER CONTINUOUS AT $(x, t) \in Q$ (in the parabolic sense) if

$$\sup_{(y,s) \in Q \setminus \{(x,t)\}} \frac{|u(y, s) - u(x, t)|}{|y - x|^\alpha + |t - s|^{\frac{\alpha}{2}}} < \infty.$$

A function $u : Q \rightarrow \mathbb{R}$ is then called α -HÖLDER CONTINUOUS if it is α -Hölder continuous at all points, and UNIFORMLY α -HÖLDER

¹ "Parabolic" openness may be defined in terms of the heat balls. We will mostly consider cylindrical sets, $Q = \Omega \times I$, where I is an interval; these are open when Ω is open (in the usual sense) and I is open at its left-endpoint (and may or may not include its right endpoint).

CONTINUOUS if

$$[u]_{\mathcal{C}^\alpha(Q)} \doteq \sup_{((x,t),(y,s)) \in Q \times Q \setminus \{(X,X): X \in Q\}} \frac{|u(y,s) - u(x,t)|}{|y-x|^\alpha + |t-s|^{\frac{\alpha}{2}}} < \infty.$$

Observe that, for each $\alpha \in (0, 1]$, the sets $C^\alpha(\Omega)$ resp. $\mathcal{C}^\alpha(Q)$ of α -Hölder continuous functions on Ω resp. Q form linear spaces under pointwise addition and scalar multiplication. We equip $C^\alpha(\Omega)$ with the norm²

$$|u|_{C^\alpha(\Omega)} \doteq |u|_{C^0(\Omega)} + [u]_{C^\alpha(\Omega)}$$

and $\mathcal{C}^\alpha(Q)$ with the norm

$$|u|_{\mathcal{C}^\alpha(Q)} \doteq |u|_{C^0(Q)} + [u]_{\mathcal{C}^\alpha(Q)}.$$

Similarly, we equip the linear space $C^{k,\alpha}(\Omega)$ of functions u on Ω all of whose partial derivatives $D^\beta u$ of order $|\beta| \leq k$ are uniformly α -Hölder continuous with the norm

$$|u|_{C^{k,\alpha}(\Omega)} \doteq |u|_{C^k(\Omega)} + \sum_{|\beta|=k} [D^\beta u]_{C^\alpha(\Omega)},$$

where we recall that

$$|u|_{C^k(\Omega)} \doteq \sum_{|\beta| \leq k} \sup_{\Omega} |D^\beta u|,$$

and the linear space $\mathcal{C}^{k,\alpha}(Q)$ of functions u on $Q \subset \mathbb{R}^n \times \mathbb{R}$ whose partial derivatives $D^\beta \partial_t^j u$ of order $|\beta| + 2j \leq k$ are uniformly α -Hölder continuous with the norm

$$|u|_{\mathcal{C}^{k,\alpha}(Q)} \doteq |u|_{\mathcal{C}^k(Q)} + \sum_{|\beta|+2j=k} [D^\beta \partial_t^j u]_{\mathcal{C}^\alpha(Q)},$$

where

$$|u|_{\mathcal{C}^k(Q)} \doteq \sum_{|\beta|+2j \leq k} \sup_Q |D^\beta \partial_t^j u|$$

The spaces $C^{k,\alpha}(\Omega, \mathbb{R}^m)$ and $\mathcal{C}^{k,\alpha}(Q, \mathbb{R}^m)$ are defined in the same manner (using vector norms where necessary). The spaces $C^{k,\alpha}(\bar{\Omega}, \mathbb{R}^m)$ and $\mathcal{C}^{k,\alpha}(\bar{Q}, \mathbb{R}^m)$ can be defined as for Ω and Q , respectively, and in this case form Banach spaces (see the Exercises).

The primary reason for the introduction of Hölder norms is that a sequence of functions u_j which is bounded in the C^k topology typically loses a derivative in passing to the limit (along a subsequence) by way of the Arzelà–Ascoli theorem. But if the sequence is bounded in $C^{k,\alpha}$, $\alpha > 0$, then it subconverges in the $C^{k,\beta}$ topology for every $\beta < \alpha$ (and in fact the limit will lie in $C^{k,\alpha}$). That is, we only need to give up an (arbitrarily small) *fraction* of a derivative. See Exercise 5.

² Note that $[\cdot]_{C^\alpha(\Omega)}$ is not a norm, since it vanishes on constant functions. Moreover, we allow $|u|_{C^\alpha(\Omega)} = \infty$, though $|u|_{C^\alpha(\Omega')} < \infty$ for every $u \in C^\alpha(\Omega)$ and $\Omega' \Subset \Omega$.

Elliptic equations

We begin by treating elliptic equations. Our first result provides a Hölder estimate for the Hessian of a solution to a linear elliptic equation away from the boundary of the domain.

Theorem 1 (Interior Schauder). *There exists $C = C(n, \lambda, \Lambda, \alpha, \rho, R) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying $\text{diam } \Omega \leq R$ and let $a \in C^\alpha(\Omega, S^{n \times n})$, $b \in C^\alpha(\Omega, \mathbb{R}^n)$, $c \in C^\alpha(\Omega)$ be coefficients satisfying*

$$a^{ij} \geq \lambda \delta^{ij} \text{ in } \Omega, \quad \lambda > 0$$

and

$$|a|_{C^\alpha(\Omega, S^{n \times n})} + |b|_{C^\alpha(\Omega, \mathbb{R}^n)} + |c|_{C^\alpha(\Omega)} \leq \Lambda, \quad \Lambda < \infty.$$

Suppose that $u \in C^{2,\alpha}(\Omega)$ satisfies

$$-(a^{ij}u_{ij} + b^i u_i + cu) = f.$$

If $f \in C^\alpha(\Omega)$, then, given any $\Omega' \Subset \Omega$ satisfying $\text{dist}(\Omega', \partial\Omega) \geq \rho$,

$$|u|_{C^{2,\alpha}(\Omega')} \leq C \left(|u|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)} \right).$$

We first prove a basic Hölder estimate for the Hessian of C^2 solutions to the Poisson equation on $\mathbb{R}^n$, from which everything will follow.

Lemma 2. *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{C^\alpha(\mathbb{R}^n)} \leq C[\Delta u]_{C^\alpha(\mathbb{R}^n)}$$

for every $u \in C^{2,\alpha}(\mathbb{R}^n)$.

Proof. The proof proceeds by *reductio ad absurdum*. So suppose, contrary to the claim, that there exists, for each $\ell \in \mathbb{N}$, some $u_\ell \in C^{2,\alpha}(\mathbb{R}^n)$ such that $\Delta u_\ell \in C^\alpha(\mathbb{R}^n)$ and yet

$$[D^2u_\ell]_{C^\alpha} > \ell[\Delta u_\ell]_{C^\alpha},$$

where $[\cdot]_{C^\alpha} \doteq [\cdot]_{C^\alpha(\mathbb{R}^n)}$. If we set $v_\ell \doteq \lambda_\ell u_\ell$, where $\lambda_\ell \doteq [D^2u_\ell]_{C^\alpha}^{-1}$, then

$$[D^2v_\ell]_{C^\alpha} = 1 \text{ and } [\Delta v_\ell]_{C^\alpha} \leq \ell^{-1}.$$

By the pigeonhole principle, we can find $i, j, k \in \{1, \dots, n\}$ such that, for infinitely many ℓ , there exist $x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$ for which

$$\frac{|D_i D_j v_\ell(x_\ell + h_\ell e_k) - D_i D_j v_\ell(x_\ell)|}{h_\ell^\alpha} \geq \frac{1}{2n^3}.$$

We now rescale once more: if we set

$$w_\ell(x) \doteq h_\ell^{-2-\alpha} v_\ell(x_\ell + h_\ell x) + p_\ell(x),$$

where p_ℓ is a quadratic polynomial, then

$$[D^2 w_\ell]_{C^\alpha} \leq 1, \quad [\Delta w_\ell]_{C^\alpha} \leq \ell^{-1}, \quad \text{and} \quad |D_i D_j w_\ell(e_k) - D_i D_j w_\ell(0)| \geq \frac{1}{2n^3}.$$

We choose p_ℓ so that w_ℓ vanishes to second order at the origin,

$$w_\ell(0) = 0, \quad Dw_\ell(0) = 0 \quad \text{and} \quad D^2 w_\ell(0) = 0.$$

Applying the Arzelà–Ascoli theorem, we can find a subsequence of the functions w_ℓ which converges locally uniformly in $C^2(\mathbb{R}^n)$ to a harmonic function w satisfying

$$D^2 w(0) = 0 \quad \text{and} \quad [D^2 w]_{C^\alpha} \leq 1, \tag{1}$$

and yet

$$|D^2 w(e_k)| \geq \frac{1}{2n^3}.$$

But this is absurd: since w is harmonic, (1) and the derivative estimates for harmonic functions,

$$\sup_{B_r^n(x)} |D^k u| \leq C(n, k) r^{-k} \sup_{\partial B_r^n(x)} |u|,$$

imply that $D^2 w \equiv 0$. □

We now prove Theorem 1 by *freezing the coefficients*, introducing cutoff functions, and applying Lemma 2. Note that, by a linear change of coordinates, Lemma 2 holds with Δ replaced by $a^{ij} D_i D_j$ for any positive definite $a \in S^{n \times n}$, with the constant C now depending on a (see Exercise 7).

Proof of Theorem 1. We will only prove the claim for $\Omega = B_2$ and $\Omega' = B_1$. Once this is proved, a simple scaling argument yields the claim for $\Omega = B_{2r}$ and $\Omega' = B_r$ for any $r > 0$. A covering argument then yields the claim for general domains Ω and $\Omega' \Subset \Omega$.

Fix any $x_0 \in B_1$ and set $a_0 \doteq a(x_0)$. In order to apply Lemma 2, we multiply u with a cutoff function η with support in the ball B_r of radius r about x_0 , with $r \leq 1$. Observe that $v \doteq \eta u$ satisfies

$$-a_0^{ij} v_{ij} = Lv + (a^{ij} - a_0^{ij}) v_{ij} + b^i v_i + cv \quad \text{in } B_r,$$

where

$$L \doteq -(a^{ij} D_i D_j + b^i D_i + c).$$

We may estimate

$$\begin{aligned} |(a^{ij} - a_0^{ij}) v_{ij}|_{C^\alpha(B_r)} &\leq [D^2 v]_{C^\alpha(B_r)} |a - a_0|_{C^0(B_r)} + [a - a_0]_{C^\alpha(B_r)} |D^2 v|_{C^0(B_r)} \\ &\leq \Lambda r^\alpha [D^2 v]_{C^\alpha(B_r)} + \Lambda |D^2 v|_{C^0(B_r)}. \end{aligned} \tag{2}$$

Thus, applying Lemma 2 (in conjunction with a coordinate transformation and a rescaling),

$$\begin{aligned}
[D^2v]_{C^\alpha(B_r)} &= [D^2v]_{C^\alpha(\mathbb{R}^n)} \\
&\leq C[a_0^{ij}v_{ij}]_{C^\alpha(\mathbb{R}^n)} \\
&= C[a_0^{ij}v_{ij}]_{C^\alpha(B_r)} \\
&= C[Lv + (a^{ij} - a_0^{ij})v_{ij} + b^i v_i + cv]_{C^\alpha(B_r)} \\
&\leq C[Lv]_{C^\alpha(B_r)} + [(a^{ij} - a_0^{ij})v_{ij}]_{C^\alpha(B_r)} + [b^i v_i]_{C^\alpha(B_r)} + [cv]_{C^\alpha(B_r)},
\end{aligned}$$

where $C = C(n, \alpha, \lambda, \Lambda)$ (i.e. C depends only on n, λ, Λ and α). Applying (2) and choosing r sufficiently small, we may absorb the first term on the right of (2). This yields an estimate of the form

$$[D^2v]_{C^\alpha(B_r)} \leq C([Lv]_{C^\alpha(B_r)} + |v|_{C^2(B_r)}),$$

where $C = C(n, \alpha, \lambda, \Lambda, \rho)$.

Now, if we arrange that $\eta \equiv 1$ in $B_{\sigma r}$, where $\sigma < 1$, then

$$[D^2u]_{C^\alpha(B_{\sigma r})} = [D^2v]_{C^\alpha(B_{\sigma r})} \leq [D^2v]_{C^\alpha(B_r)}.$$

On the other hand, we may crudely estimate

$$\begin{aligned}
[Lv]_{C^\alpha(B_r)} &= [\eta Lu + (a^{ij}\eta_{ij} + b^i\eta_i)u + a^{ij}\eta_i u_j]_{C^\alpha(B_r)} \\
&\leq C(|f|_{C^\alpha(B_r)} + |u|_{C^2(B_r)}),
\end{aligned}$$

where $C = C(n, \lambda, \Lambda, \eta)$, and

$$|v|_{C^2(B_r)} \leq C|u|_{C^2(B_r)},$$

where $C = C(n, \eta)$. Thus, by covering B_1 with suitable balls of radius r and choosing η appropriately, we obtain

$$|u|_{C^{2,\alpha}(B_1)} \leq C(|f|_{C^\alpha(B_{3/2})} + |u|_{C^2(B_{3/2})}), \quad (3)$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

The final step is to replace the C^2 -norm on the right hand side by the C^0 norm using the following interpolation inequality³:

$$|D^2w|_{C^0(B_1)} \leq \varepsilon[D^2w]_{C^\alpha(B_1)} + C_\varepsilon|w|_{C^0(B_1)}, \quad (4)$$

where $C_\varepsilon = C_\varepsilon(n, \alpha, \varepsilon)$. Though this does not look good enough, since the norms on the right are over the larger ball $B_{3/2}$, it can actually still be achieved by exploiting the fact that (3) holds for *all* solutions to equations satisfying the hypotheses.

Fix any $y \in B_2$, set $\sigma = \frac{1}{3} \text{dist}(y, \partial B_2)$, and consider the function $\tilde{u} : B_2 \rightarrow \mathbb{R}$ defined by

$$\tilde{u}(x) \doteq u(y + \sigma x).$$

³ A much simplified version of this inequality is established in the Exercises.

Observe that

$$\tilde{L}\tilde{u}(x) = \sigma^2 Lu(y + \sigma x),$$

where

$$\tilde{L}(x) \doteq a^{ij}(y + \sigma x)D_i D_j + \sigma b^i(y + \sigma x)D_i + \sigma^2 c(y + \sigma x).$$

Since $y + \sigma B_2 \subset B_2$ and $\sigma \leq 1$, the estimate (3) yields

$$\begin{aligned} |\tilde{u}|_{C^{2,\alpha}(B_1)} &\leq C([\tilde{L}\tilde{u}]_{C^\alpha(B_2)} + |\tilde{u}|_{C^2(B_2)}) \\ &\leq C([Lu]_{C^\alpha(B_2)} + |u|_{C^0(B_2)} + \sigma^2 |D^2 u|_{C^0(B_{2\sigma}(y))}), \end{aligned} \quad (5)$$

where $C = C(n, \alpha, \lambda, \Lambda)$, and we used the standard interpolation inequality to estimate

$$2\sigma |Du|_{C^0(B_2)} \leq C(n) (|u|_{C^0(B_2)}^2 + \sigma^2 |D^2 u|_{C^0(B_2)}^2).$$

Applying the interpolation inequality (4), we may estimate

$$\begin{aligned} \frac{1}{9} \text{dist}(y, \partial B_2)^2 |D^2 u(y)| &\leq \sigma^2 |D^2 u|_{C^0(B_{2\sigma}(y))} \\ &= |D^2 \tilde{u}|_{C^0(B_1)} \\ &\leq \varepsilon |\tilde{u}|_{C^{2,\alpha}(B_1)} + C_\varepsilon |\tilde{u}|_{C^0(B_1)}. \end{aligned}$$

Since

$$\begin{aligned} \sigma^2 \sup_{x \in B_{2\sigma}(y)} |D^2 u(x)| &\leq \sup_{x \in B_{2\sigma}(y)} \text{dist}(x, \partial B_2) |D^2 u(x)| \\ &\leq \sup_{x \in B_2} \text{dist}(x, \partial B_2)^2 |D^2 u(x)|^2 \\ &\doteq Q, \end{aligned}$$

recalling (5) and choosing $\varepsilon = \frac{1}{18C}$, we obtain

$$\frac{1}{9} \text{dist}(y, \partial B_2)^2 |D^2 u(y)| \leq C (|Lu|_{C^\alpha(B_2)}^2 + |u|_{C^0(B_2)}) + \frac{1}{18} Q.$$

Since y was arbitrary, we conclude that

$$\frac{1}{4} |D^2 u|_{C^0(B_{3/2})} \leq Q \leq 9C (|f|_{C^\alpha(B_2)}^2 + |u|_{C^0(B_2)}).$$

This completes the proof. $\square$

We next establish an estimate in the neighbourhood of any boundary point (with additional dependence on the boundary data). Together, the interior and boundary estimates then yield a global estimate.

Recall that an open set $\Omega \subset \mathbb{R}^n$ is said to be of class $C^{k,\alpha}$, if each point p of its boundary $\partial\Omega$ admits a neighbourhood U in $\mathbb{R}^n$ and an injective map $\phi : U \rightarrow \mathbb{R}^n$ onto $V \subset \mathbb{R}^n$ such that

$$\phi(\Omega \cap U) \subset \mathbb{R}_+^n, \quad \phi(\partial\Omega \cap U) \subset \partial\mathbb{R}_+^n, \quad \phi \in C^{k,\alpha}(U) \quad \text{and} \quad \phi^{-1} \in C^{k,\alpha}(V),$$

where $\mathbb{R}_+^n$ is the UPPER HALF-SPACE $\mathbb{R}^{n-1} \times (0, \infty)$ in $\mathbb{R}^n$. The pair (ϕ, U) is called a CHART for Ω . If $k \geq 1$, then, by the implicit function theorem, $\partial\Omega$ may also be represented as a graph over its tangent plane (see Exercise 8).

Theorem 3 (Global Schauder). *There exists $C = C(n, \lambda, \Lambda, \alpha, \Omega) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^{2,\alpha}$ and let $a \in C^\alpha(\Omega, S^{n \times n})$, $b \in C^\alpha(\Omega, \mathbb{R}^n)$, $c \in C^\alpha(\Omega)$ be coefficients satisfying*

$$a^{ij} \geq \lambda \delta^{ij} \text{ in } \Omega, \quad \lambda > 0$$

and

$$|a|_{C^\alpha(\Omega, S^{n \times n})} + |b|_{C^\alpha(\Omega, \mathbb{R}^n)} + |c|_{C^\alpha(\Omega)} \leq \Lambda, \quad \Lambda < \infty.$$

Suppose that $u \in C^{2,\alpha}(\overline{\Omega})$ satisfies

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

If $f \in C^\alpha(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, then

$$|u|_{C^{2,\alpha}(\Omega)} \leq C \left(|u|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)} + |\varphi|_{C^{2,\alpha}(\Omega)} \right).$$

The key estimate in this case is a Hölder estimate for the Hessian of C^2 solutions to Poisson's equation in the halfspace $\mathbb{R}_+^n \doteq \mathbb{R}^{n-1} \times [0, \infty)$.

Lemma 4. *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{C^\alpha(\mathbb{R}_+^n)} \leq C \left([\Delta u]_{C^\alpha(\mathbb{R}_+^n)} + [D^2u|_{\partial\mathbb{R}_+^n}]_{C^\alpha(\partial\mathbb{R}_+^n)} \right)$$

for every $u \in C^{2,\alpha}(\mathbb{R}_+^n)$.

Proof. It suffices to prove the claim in the case $D^2u|_{\partial\mathbb{R}_+^n} \equiv 0$. Indeed, in the general case, we set, for each $(x, x_n) \in \mathbb{R}^{n-1} \times [0, \infty)$,

$$\varphi(x, x_n) \doteq u|_{\partial\mathbb{R}_+^n}(x)$$

Observe that the function $v \doteq u - \varphi$ vanishes on $\partial\mathbb{R}_+^n$. So, assuming the theorem holds in this case, we find that

$$\begin{aligned} [D^2u]_{C^\alpha(\mathbb{R}_+^n)} &= [D^2v + D^2\varphi]_{C^\alpha(\mathbb{R}_+^n)} \\ &\leq [D^2v]_{C^\alpha(\mathbb{R}_+^n)} + [D^2\varphi]_{C^\alpha(\mathbb{R}_+^n)} \\ &\leq C[\Delta v]_{C^\alpha(\mathbb{R}_+^n)} + [D^2\varphi]_{C^\alpha(\mathbb{R}_+^n)} \\ &= C[\Delta u - \Delta\varphi]_{C^\alpha(\mathbb{R}_+^n)} + [D^2\varphi]_{C^\alpha(\mathbb{R}_+^n)} \\ &\leq C' \left([\Delta u]_{C^\alpha(\mathbb{R}_+^n)} + [D^2\varphi]_{C^\alpha(\mathbb{R}_+^n)} \right) \\ &= C' \left([\Delta u]_{C^\alpha(\mathbb{R}_+^n)} + [D^2\varphi]_{C^\alpha(\partial\mathbb{R}_+^n)} \right), \end{aligned}$$

where $C' = C'(n, \alpha)$.

So we may assume that $D^2u|_{\partial\mathbb{R}_+^n} \equiv 0$. The proof in this case proceeds, as in Lemma 4, by *reductio ad absurdum*. So suppose, contrary to the claim, that there exists, for each $\ell \in \mathbb{N}$, some $u_\ell \in C^2(\mathbb{R}_+^n)$ such that $\Delta u_\ell \in C^\alpha(\mathbb{R}_+^n)$ and $D^2u_\ell|_{\partial\mathbb{R}_+^n} \equiv 0$, and yet

$$[D^2u_\ell]_{C^\alpha} > \ell [\Delta u_\ell]_{C^\alpha},$$

where $[\cdot]_{C^\alpha} \doteq [\cdot]_{C^\alpha(\mathbb{R}_+^n)}$. If we set $v_\ell \doteq \lambda_\ell u_\ell$, where $\lambda_k \doteq |D^2u_\ell|_{C^\alpha}^{-1}$, then

$$[D^2v_\ell]_{C^\alpha} = 1 \text{ and } [\Delta v_\ell]_{C^\alpha} \leq \ell^{-1}.$$

By the pigeonhole principle, we can find $i, j, k \in \{1, \dots, n\}$ such that, for infinitely many ℓ , there exist $x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$ for which

$$\frac{|D_i D_j v_\ell(x_\ell + h_\ell e_k) - D_i D_j v_\ell(x_\ell)|}{h_\ell^\alpha} \geq \frac{1}{2n^3}.$$

After passing to a subsequence, we may arrange that either

$$h_\ell^{-1} \text{dist}(x_\ell, \partial\mathbb{R}_+^n) \rightarrow \infty \text{ as } \ell \rightarrow \infty,$$

or

$$\sup_\ell h_\ell^{-1} \text{dist}(x_\ell, \partial\mathbb{R}_+^n) < \infty.$$

In the first case, we may proceed exactly as in the proof of Lemma 2: by rescaling and translating appropriately, and adding a suitable quadratic, we obtain a sequence of functions w_ℓ with $|D^2v_\ell(0)| \geq \frac{1}{2n^3}$ which converges in the C^2 topology on compact subsets of $\mathbb{R}^n$ to the zero function, an impossibility.

If, instead, $\gamma \doteq \sup_\ell h_\ell^{-1} \text{dist}(x_\ell, \partial\mathbb{R}_+^n) < \infty$, then we can find, for all sufficiently large ℓ , some $z_\ell \in \partial\mathbb{R}_+^n$ such that $\text{dist}(x_\ell, z_\ell) \leq 2\gamma h_\ell$. Consider, for each ℓ , the function

$$w_\ell(x) \doteq h_\ell^{-2-\alpha} v_\ell(z_\ell + h_\ell x) + p_\ell(x),$$

where p_ℓ is a quadratic polynomial. Note that w_ℓ is still defined on the halfspace $\mathbb{R}_+^n$ and that the point $y_\ell \doteq h_\ell^{-1}(x_\ell - z_\ell)$ lies, for all ℓ , in a fixed compact set (the ball $\overline{B}_{2\gamma} \cap \mathbb{R}_+^n$). Moreover,

$$[D^2w_\ell]_{C^\alpha} \leq 1, \quad [\Delta w_\ell]_{C^\alpha} \leq \ell^{-1}, \quad \text{and} \quad |D_i D_j w_\ell(y_\ell + e_k) - D_i D_j w_\ell(y_\ell)| \geq \frac{1}{2n^3}.$$

We again choose p_ℓ so that w_ℓ vanishes to second order at the origin. Applying the Arzelà–Ascoli theorem, we can find $y \in \overline{B}_{2\gamma} \cap \mathbb{R}_+^n$ and a subsequence of the functions w_ℓ which converges locally uniformly in $C^2(\mathbb{R}_+^n)$ to a *harmonic* limit w which satisfies

$$D^2w(0) = 0 \text{ and } [D^2w]_{C^\alpha} \leq 1, \quad (6)$$

and yet

$$|D^2w(y + e_k) - D^2w(y)| \geq \frac{1}{2n^3}.$$

But this is absurd: since w vanishes along $\partial\mathbb{R}_+^n$, it extends by reflection across $\partial\mathbb{R}_+^n$ to a harmonic function on $\mathbb{R}^n$, so the Bernstein estimates and (6) imply that $D^2w \equiv 0$. $\square$

Freezing the coefficients, introducing cutoff functions, “straightening the boundary” using charts and applying the interpolation inequality as in the proof of Theorem 1 leads to an estimate for linear equations on bounded domains in a neighbourhood of the boundary (which now depends on the boundary condition and the boundary charts). Combined with the interior estimate, this proves Theorem 3. We omit the details.

Parabolic equations

We begin once more with an interior estimate. To keep things relatively simple, we simply state the local result (for parabolic cylinders $P_r(x, t) \doteq B_r(x) \times (t - r^2, t]$).

Theorem 5 (Interior Schauder). *There exists $C = C(n, \lambda, \Lambda, \alpha, R) < \infty$ with the following property. Suppose that $a \in C^\alpha(P_{2R}(x_0, t_0), S^{n \times n})$, $b \in C^\alpha(P_{2R}(x_0, t_0), \mathbb{R}^n)$ and $c \in C^\alpha(P_{2R}(x_0, t_0))$ satisfy*

$$a^{ij} \geq \lambda \delta^{ij}, \quad \lambda > 0$$

and

$$\|a\|_{\mathcal{C}^\alpha(P_{2R}(x_0, t_0), S^{n \times n})} + \|b\|_{\mathcal{C}^\alpha(P_{2R}(x_0, t_0), \mathbb{R}^n)} + \|c\|_{\mathcal{C}^\alpha(P_{2R}(x_0, t_0))} \leq \Lambda, \quad \Lambda < \infty.$$

If

$$f \doteq u_t - (a^{ij}u_{ij} + b^k u_k + cu + f) \in \mathcal{C}^\alpha(P_{2R}(x_0, t_0)),$$

then

$$\|u\|_{\mathcal{C}^{2, \alpha}(P_R(x_0, t_0))} \leq C(\|f\|_{\mathcal{C}^\alpha(P_{2R}(x_0, t_0))} + \|u\|_{L^\infty(P_{2R}(x_0, t_0))}).$$

As in the elliptic case, we first prove a basic Hölder estimate for the Hessian of solutions to the inhomogeneous heat equation on $\mathbb{R}^n \times (-\infty, 0]$, from which everything will follow.

Lemma 6. *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{\mathcal{C}^\alpha(\mathbb{R}^n \times (-\infty, 0])} \leq C\|(\partial_t - \Delta)u\|_{\mathcal{C}^\alpha(\mathbb{R}^n \times (-\infty, 0])}$$

for every $u \in \mathcal{C}^{2, \alpha}(\mathbb{R}^n \times (-\infty, 0])$.

Proof. The idea is exactly the same as in the elliptic case: under the assumption that the claim is false, there must exist a sequence of functions $u_\ell \in \mathcal{C}^{2,\alpha}(\mathbb{R}^n \times (-\infty, 0])$ such that $(\partial_t - \Delta)u_\ell \in C^\alpha(\mathbb{R}^n \times (-\infty, 0])$ and yet

$$[D^2 u_\ell]_{\mathcal{C}^\alpha} > \ell \|(\partial_t - \Delta)u_\ell\|_{\mathcal{C}^\alpha},$$

where we use the shorthand $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^n \times (-\infty, 0])$. Parabolically rescaling by $[D^2 u_\ell]_{\mathcal{C}^\alpha}^{-1}$, we obtain a sequence of functions $v_\ell \in \mathcal{C}^{2,\alpha}$ such that

$$[D^2 v_\ell]_{\mathcal{C}^\alpha} = 1 \quad \text{and} \quad \|(\partial_t - \Delta)v_\ell\|_{\mathcal{C}^\alpha} \leq \frac{1}{\ell}.$$

By the pigeonhole principle, we can find $i, j, k \in \{1, \dots, n\}$ such that, for infinitely many ℓ , there exist $(x_\ell, t_\ell) \in \mathbb{R}^n \times (-\infty, 0]$ and $(y_\ell, s_\ell) \in \mathbb{R}^n \times (-\infty, 0]$ with $s_\ell \leq t_\ell$ for which

$$\frac{|D_i D_j v_\ell(x_\ell, t_\ell) - D_i D_j v_\ell(y_\ell, s_\ell)|}{\lambda_\ell^\alpha} \geq \frac{1}{2n^3},$$

where $\lambda_\ell = |x_\ell - y_\ell| + \sqrt{t_\ell - s_\ell}$. We now pass to a subsequence so that this holds for all ℓ , rescale by λ_ℓ , shift (x_ℓ, t_ℓ) to the spacetime origin, to obtain a sequence of functions $w_\ell \in \mathcal{C}^{2,\alpha}$ such that

$$[D^2 w_\ell]_{\mathcal{C}^\alpha} \leq 1, \quad \|(\partial_t - \Delta)w_\ell\|_{\mathcal{C}^\alpha} \leq \frac{1}{\ell}, \quad |D_i D_j w_\ell(0, 0) - D_i D_j w_\ell(z_\ell, u_\ell)| \geq \frac{1}{2n^3},$$

where $(z_\ell, u_\ell) \doteq (\lambda_\ell^{-1}(y_\ell - x_\ell), \lambda_\ell^{-2}(s_\ell - t_\ell))$. We may further arrange, by adding a suitable (spatially quadratic, temporally linear) polynomial, that

$$w_\ell(0, 0) = 0, \quad Dw_\ell(0, 0) = 0, \quad D^2 w_\ell(0, 0) = 0 \quad \text{and} \quad \partial_t w_\ell(0, 0) = 0.$$

Since $(z_\ell, u_\ell) \in \{(y, s) \in \mathbb{R}^n \times (-\infty, 0] : |y| + \sqrt{-s} = 1\}$, which is compact, we may arrange, by passing to a subsequence, that $(z_\ell, u_\ell) \rightarrow (z_\infty, u_\infty) \in \{(y, s) \in \mathbb{R}^n \times (-\infty, 0] : |y| + \sqrt{-s} = 1\}$. The Arzelà–Ascoli theorem now guarantees that, after passing to a subsequence, $w_\ell \rightarrow w_\infty \in \mathcal{C}^{2,\alpha}$ in $\mathcal{C}^{2,\beta}$ for any $\beta < \alpha$. The limit must satisfy

$$(\partial_t - \Delta)w_\infty = 0, \quad [D^2 w_\infty]_{\mathcal{C}^\alpha} \leq 1$$

and

$$w_\infty(0, 0) = 0, \quad Dw_\infty(0, 0) = 0, \quad D^2 w_\infty(0, 0) = 0 \quad \text{and} \quad \partial_t w_\infty(0, 0) = 0,$$

but

$$D_i D_j w_\infty(0, 0) \geq \frac{1}{2n^3}.$$

But this is absurd since, by the derivative estimates for the heat equation, the previous conditions guarantee that $D^2 u \equiv 0$. $\square$

Note that, by a linear change of coordinates, Lemma 6 holds with Δ replaced by $a^{ij}D_iD_j$ for any (constant) positive definite $a \in S^{n \times n}$, with the constant C now depending also on a . Theorem 5 then follows by freezing the coefficients, introducing cutoff functions, and exploiting interpolation estimates for Hölder spaces.

As in the elliptic case, a (spatial) boundary estimate may be deduced from a corresponding estimate for the heat operator in a half-space.

Lemma 7. *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{\mathcal{C}^\alpha(\mathbb{R}_+^n \times (-\infty, 0])} \leq C(\|(\partial_t - \Delta)u\|_{\mathcal{C}^\alpha(\mathbb{R}_+^n \times (-\infty, 0])} + [D^2u]_{\mathcal{C}^\alpha(\partial\mathbb{R}^n \times (-\infty, 0])})$$

for every $u \in \mathcal{C}^{2,\alpha}(\mathbb{R}_+^n \times (-\infty, 0])$.

Exercises

Exercise 1. Provide an example of

- (a) a function $u : [0, 1] \rightarrow \mathbb{R}$ which is in $C^\alpha([0, 1])$ but not in $C^\beta([0, 1])$ for any $\beta > \alpha$.
- (b) a function $u : [0, 1] \rightarrow \mathbb{R}$ which is in $C^{1,1}([0, 1])$ but not in $C^2([0, 1])$.

Exercise 2. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is continuously differentiable. Show that f is α -Hölder continuous for all $\alpha \in (0, 1]$.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Show, for any choice of $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $u, v \in C^\alpha(\Omega)$, $\alpha \in (0, 1]$, show that

$$[uv]_{C^\alpha(\Omega)} \leq \sup_{\Omega} |u| [v]_{C^\alpha(\Omega)} + \sup_{\Omega} |v| [u]_{C^\alpha(\Omega)}.$$

Exercise 5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $\alpha, \beta \in (0, 1]$ with $\beta \leq \alpha$, show that

- $C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})$.
- the inclusion map $\iota : C^\alpha(\overline{\Omega}) \rightarrow C^\beta(\overline{\Omega})$ is continuous.

- the embedding $\iota : C^\alpha(\overline{\Omega}) \rightarrow C^\beta(\overline{\Omega})$ is compact⁴ if $\beta < \alpha$.
Hint: by the Arzelà–Ascoli theorem, a bounded sequence $\{u_j\}_{j \in \mathbb{N}} \subset C^\alpha(\overline{\Omega})$ admits a subsequence which converges in the uniform topology to some limit u . Show that $v_j \doteq u_j - u$ converges to zero in the β -Hölder topology.

⁴ That is, bounded subsets of $C^\alpha(\overline{\Omega})$ are relatively compact in $C^\beta(\overline{\Omega})$.

Exercise 6. Let u be a smooth function on $\mathbb{R}$ with finite $C^{1,\alpha}$ norm.

- (a) Given $x \in \mathbb{R}$, show that, for any $h > 0$, there exists some $x_0 \in \mathbb{R}$ such that

$$|x - x_0| \leq h \text{ and } |u'(x_0)| \leq \frac{2}{h} \sup_{\mathbb{R}} |u|$$

- (b) Deduce that

$$\sup_{\mathbb{R}} |u'| \leq \varepsilon \sup_{x \neq y} \frac{|u'(x) - u'(y)|}{|x - y|^\alpha} + \frac{2}{\varepsilon^{\frac{1}{\alpha}}} \sup_{\mathbb{R}} |u|$$

for any $\varepsilon > 0$.

- (c) Deduce that

$$\sup_{\mathbb{R}} |u''| \leq \varepsilon \sup_{x \neq y} \frac{|u''(x) - u''(y)|}{|x - y|^\alpha} + C(\varepsilon, \alpha) \sup_{\mathbb{R}} |u|$$

for any $\varepsilon > 0$.

Exercise 7. Given a positive definite $a \in S_+^{n \times n}$, suppose that $u \in C^2(\mathbb{R}^n)$ satisfies

$$-a^{ij}u_{ij} = 0.$$

Find a linear change of coordinates $A \in GL(\mathbb{R}^n)$ such that $x \mapsto u(Ax)$ is harmonic.

Exercise 8. Suppose that $\Omega \subset_{\text{open}} \mathbb{R}^n$ is of class $C^{k,\alpha}$ for some $k \geq 1$. The TANGENT SPACE (OR TANGENT PLANE) $T_x\partial\Omega$ to $\partial\Omega$ at $x \in \partial\Omega$ is the affine subspace of $\mathbb{R}^n$ defined by

$$T_x\partial\Omega \doteq \{x + D_v\phi^{-1} : v \in \partial\mathbb{R}_+^n\},$$

where $\phi : U \rightarrow \mathbb{R}^n$ is some chart for Ω with $x \in U$. The OUTWARD UNIT NORMAL vector $\nu(x)$ to Ω at x is the unit normal vector to $T_x\partial\Omega$ with the property that $x + \varepsilon\nu(x) \in \mathbb{R}^n \setminus \Omega$ for all ε sufficiently small.

- (a) Show that $T_x\partial\Omega$ does not depend on the choice of chart about x .
 (b) Show that, for each $x \in \partial\Omega$, there is a function $u \in C^{k,\alpha}(T_x\partial\Omega)$ and a neighbourhood U of x in $\mathbb{R}^n$ such that

$$\Omega \cap U = \{y + r\nu(x) : y \in T_x\partial\Omega \text{ and } r < u(y)\} \cap U$$

and

$$\partial\Omega \cap U = \{y + u(y)\nu(x) : y \in T_x\partial\Omega\} \cap U.$$

Hint: You will need the implicit function theorem.

More generally, a subset $M \subset \mathbb{R}^n$ is called an EMBEDDED SUB-MANIFOLD (of dimension $\ell < n$ and class $C^{k,\alpha}$) if, for each $x \in M$, there is a neighbourhood U of x in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \rightarrow V \subset \mathbb{R}^n$ of class $C^{k,\alpha}$ such that $\phi(U \cap M) = V \cap (\mathbb{R}^\ell \times \{0\})$. If $k \geq 1$, then M admits a tangent space and unit normal at each point, and may be represented as the graph of a $C^{k,\alpha}$ function over each tangent space.